Pavol Jozef Šafárik University in Košice
Faculty of Science
Department of Theoretical Physics and Astrophysics

STATISTICAL PROPERTIES OF RANDOM FIELDS IN STOCHASTIC DYNAMICS AND FULLY DEVELOPED TURBULENCE

RNDr. **TOMÁŠ LUČIVJANSKÝ**

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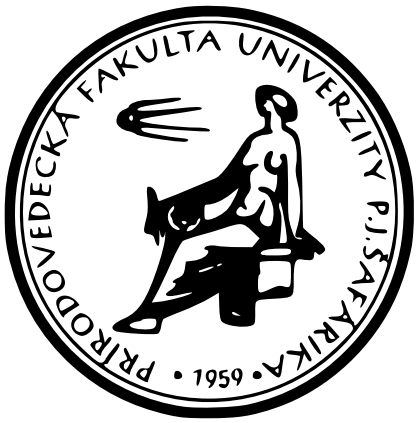
Pavol Jozef Šafárik University in Košice
Faculty of Science
Institute of Physics



Statistical Properties of Random Fields in Stochastic Dynamics and Fully Developed Turbulence

Dissertation

Author of the dissertation: RNDr. Tomáš Lučivjanský
Supervisor of the dissertation: Doc. RNDr. Michal Hnatič, DrSc.

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I declare this dissertation was written on my own, with the only help provided by my supervisor and the referred-to literature.

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In Košice 30th August, 2012 Signature

ABSTRAKT

Anihilačné reakcie dvoch rovnakých typov molekúl, ktoré sú vystavené vonkajšiemu advektívne poľu, predstavujú jeden z najjednoduchších modelov nerovnovážnej štatistickej fyziky. Pre takýto typ reakcií, odohrávajúcich sa v nízkych priestorových dimenziách, nie je popis založený na rýchlostnej rovnici adekvátny a je nutné vziať do úvahy vplyv koncentračných fluktuácii. Použitím poruchovej renormaližačnej grupy je možné študovať vplyv náhodného rýchlostného poľa na kinetiku anihilačnej reakcie typu $A + A \rightarrow \varnothing$ v oblasti jej kritickej dimenzie $d_c = 2$. Pre generovanie rýchlostného poľa je použitá stochastická Navierova-Stokesova rovnica, Kraichnanov model s konečným korelačným časom a Kraichnanov model so zarátaním kompresibility. Použitím Doiovej metódy druhého kvantovania je stochastický reakčno-difúzny problém možné namapovať na kvantovopoľový model, ktorého asymptotické riešenia môžu byť analyzované spomínanou technikou renormalizačnej grupy. Anihilačná reakcia je študovaná v blízkosti dimenzie 2 pomocou dvojparametrového rozvoja v (ϵ, Δ), kde ϵ je odklon od Kolmogorovského škálovania a Δ je odklon od priestorovej dimenzie 2. Všetky relevantné veličiny sú vypočítané do druhého rádu poruchovej teórie a dlhočasové správanie sa stochastického systému je určené. Taktiež do prvého rádu poruchovej teórie je študovaný efekt kompresiblity na priebeh anihilačnej reakcie prostredníctvom zovšeobecneného Kraichnanovho modelu.

Efekty náhodnych zdrojov a únikov na reakčnú kinetiku sú skúmané prostredníctvom riadiacej rovnice v poľovo-teoretickej formulácii, ktorá je odvodená pomocou metódy druhého kvantovania v Doiovom formalizme. Je demonštrované, že náhodné zdroje a únikové mechanizmy maj́u dôležitý vplyv na asymptotické vlastnosti modelu a dve triedy univerzality sú identifikované pomocou škálovacej analýzy.

Použitím poruchovej renormalizačnej grupy je analyzovaný vplyv náhodného Gaussovského poľa rýchlosti na kritické správanie sa anizotrópného perkolačného procesu. Poľovo-teoretická renormalizačná grupa je aplikovaná za účelom získania informácie o možnom veľko-škálovom chovaní sa systému. Pevné body a zodpovedajúce oblasti stability sú určené v hlavnom stupni poruchovej teórie.

Kľúčové slová: reakčno-difúzne problémy, turbulentná advekcia, poľovo-teoretický model, poruchová renormalizačná grupa

ABSTRACT

Single-species annihilation reactions, where the reactants are influenced by an external advective field, are one of the simplest examples of irreversible statistical systems. For this type of reaction in low space dimensions the usual description by means of kinetic rate equation is not sufficient and the effect of density fluctuations must be taken into the account. Using perturbative renormalization group we study the influence of a random velocity field on the kinetics of a single-species annihilation reaction $A + A \rightarrow \varnothing$ at and below its critical dimension $d_c = 2$. In order to generate the velocity field stochastically forced Navier-Stokes equation and Kraichnan model with finite time correlations is used. We use the second-quantization formalism of Doi to cast the stochastic problem into a field-theoretic form. The annihilation reaction is analysed near two dimensions by means of two-parameters expansion ϵ, Δ. ϵ is a deviation from Kolmogorov scaling and Δ is a deviation from the space dimension 2. All of the relevant quantities are evaluated to the second order of the perturbation scheme and the long-time asymptotic behaviour of stochastic system under consideration is estimated. Also the effect of compressibility on the time evolution of annihilation process was studied with the use of a generalised Kraichnan model to the first order approximation.

The effects of random sources and sinks on reaction kinetics in the master-equation description have been investigated in the framework of a field-theoretic model, obtained by the "second quantization" à la Doi. It has been demonstrated that random sources and sinks have a significant effect on the asymptotic behaviour of the model and two universality classes for their description have been identified by the scaling analysis. Results are compared with the Langevin-equation description of the same process.

Using perturbative renormalization group also the influence of a random Gaussian velocity field on the critical behaviour of directed bond percolation process is studied. The field-theoretic renormalization group is applied to obtain information about possible large scale behavior. Fixed points with corresponding regions of stability are evaluated to the leading order in the perturbation scheme.

Keywords: reaction-diffusion problems, turbulent advection, field-theoretic model, perturbative renormalization group

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Contents

List of Figures

List of Tables

1 INTRODUCTION

1.1 MOTIVATION

Statistical physics forms fundamental cornerstone of the modern science. Since its foundation as scientific discipline in the works of Gibbs and Boltzmann it has evolved to a great depth both in scope and rigorousness. At present, methods primarily devoted to the study of physical systems, are applied in such diverse scientific fields as chemistry, biology, economics, sociology or computer science. Such success might be explained by the generality of the fundamental laws of statistical physics and genuine appearance of systems with great resemblance to the models studied in physics. In general they could be characterised by large number of entities (atoms,spins and so on) that interacts with each other. It is important to realize how large is this number. For instance only one mole of ordinary matter under normal circumstances contains as many as 10^{23} atoms or molecules. This is incredibly large value beyond human ability to imagine. The rigourous treatment of such system in terms of particle dynamics (via classical Euler-Lagrange equations or Schrödinger equation in the quantum case) is apparently meaningless. Nonetheless it is experimental fact that under the stationary boundary conditions all systems tend to evolve to the equilibrium state, that is characterised by constant value of its macroscopic characteristics. In more precise sense it is formulated through the second thermodynamic law, that also identifies equilibrium with the most unordered (equivalent to the most probable) state under given external conditions. In the majority of cases encountered in equilibrium physics it is possible to use an approximation in which interactions or fluctuations between microscopic constituents of the model can be neglected or treated as a small perturbation to the ideal situation. Ideal gas and van der Waals model, that takes into account finite size of gas particles, are the famous examples.

However, there also exist situations, where such approach is not appropriate. Theory of critical phenomena [1, 2], which deals with the second-order phase transitions in macroscopic systems, is a well-known representative. It is observed, e.g. in liquid-vapour transition, $\lambda-$transition in superfluid helium or various magnetic transitions between paramagnetic and ferromagnetic phases. Characteristic feature of these transitions is appearance of strong fluctuations and correlations between underlying constituents (atoms or spins). Parameter that could be used for quantitative description of correlations is a correlation length. Broadly speaking it represents the average distance to which atoms "feel" each other or behave cooperatively. In equlibrium situations it is of atomic order,

which explains why atomic system usually could be considered as consisting of effectively non-interacting atoms. The first attempt to tackle problem of phase transitions was based on the use mean field theory. Loosely speaking it presumes that correlations between microscopic stochastic variables can be treated perturbatively [3]. In the sense of central limit theorem then deviations from the ideal Gaussian behavior can be constucted (in quantum field theory this procedure is also known as Wick theorem). In most case some equation of state for macroscopic quantities can then be directly obtained. In the situations, when the system is far from the critical region, the correlation length is very small and can be effectively neglected. There is no obvious discrepancy with the experimental data. However it turns out, that this approach exhibits large quantitative differences in the vicinity of critical point. The problem is that here the correlations are very large, in fact directly at transition point correlation length is divergent. Therefore the perturbation theory is not applicable anymore (coupling constant is very large and by no means treated as a small quantity). But divergent correlation length brings another important player into the game - scale invariance, which is not usual invariance. Rather it is a kind of dynamically generated (emergent) symmetry. Mathematically it is an invariance with respect to the special class of transformations, that account for a change in scale at which physical system is studied. Divergent correlation length means, that there is no special scale and system looks and behaves at every scale in the same way. Well-known observation in physics is that symmetry can have a profound influence on the properties of the physical system. In terms of scale invariance the experimentally observed power laws for functional dependence of various termodynamic functions can be explained and also the various relations between critical indices can be quantitavely estimated. Another important property of the second order phase transition is *universality*. In a simple formulation it says, that the behavior of the system near its critical point is fully determined by the universal quantities - dimension of space, number of components of order parameter, symmetry constraints , that are not characteristic for only one system, but whole class of systems and also states that universal quantities does not depend on the model-dependent parameters - coupling constants etc. Thus in the detail very different physical systems as strongly anistropic magnetic material (as Ising model) and liquid can have the same properties near critical point. One only has to know few very general information to classify physical system according to its critical behavior into some universality class, wherein all the systems behave in the same way.

Similar situation to the phase transitions was also observed in the particle physics. Fundamental theory of matter quantum chromodynamics has at average energies cou-

pling constants of order unity. So it also represents strong-coupling system. For them renormalization Group (RG) technique was developed mainly in the works of Stueckelberg, Petermann, Gell-Man, Low, N.N. Bogolyuov, D. V. Shirkov [4]-[9] and others. It is interesting that there is a great intrinsic similarity between quantum models in particle physics and statistical field models. In terms of path integral representation it is possible to show the correspondence between a classical model in $d+1$-space dimensions and a quantum model in d-dimesions [10].

RG method not only leads to the quantitative description of the behavior near critical points, it also provides a new framework in which aforementioned scale invariance and universality is naturally explained. At the present the use of RG method in equilibrium statistical physics is well established and represents an important theoretical tool.

Contrary to the equilibrium physics there are only few rigorous results for the case of non-equilibrium systems. Some of their properties are reminiscent of the equilibrium systems and thus it seems natural to apply RG method for them. However, there exists also fundamental differences between them. Non-equilibrium systems might be divided [11] into two broad classes:

(a) systems with hermitian Hamiltonian, whose stationary states are described by Gibbs-Boltzmann distribution. Nevertheless at beginning they happen to be in a state far from the stationary (equilibrium) state. Dynamic descriptions of such systems are obtained directly from static formulations. Examples: Landau-Ginzburg equation for time evolution of local magnetization, kinetic Ising model, models A-H for various models of critical dynamics [12]. All these equations are specific realisations of universal Langevin equation [1].

(b) systems without hermitian Hamiltonian or without Hamiltonian description at all, which in general does not need to have a stationary state. The detailed balance condition is not satisfied for them, which implies that Einstein relation between thermal fluctuations and friction forces can not be stated. The typical examples of such systems are: fluid in turbulent state, irreversible chemical processes, surface growing models etc.

Other approaches to such systems has to be used as via quite general stochastic differential equation, which can be considered as an extension of Langevin equation or using master equation [13]. Former equation is suggested for some macroscopic quantity and because of negligence of microscopic structure random force has to be introduced. Then according to some phenomenological observations proper properties

of random force has to be chosen. The latter approach is probably more fundamental, but also more difficult to handle.

It is a well-known fact [1] that the failure of Landau theory of the second-order phase transition lies in the assumption of analycity of energy functional $F = F(\varphi)$, where $\varphi(x)$ is an order parameter configuration. The fluctuation theory of phase transitions takes as a fundamental quantity not the average value $\alpha = \langle\varphi(x)\rangle$, but the random field $\varphi(x)$ itself. The difference with mean field theory is to take Landau functional as fundamental Hamiltonian instead of exact microscopic model. To calculate some physical quantities one has to perform average over all configurations $\varphi(x)$. Mathematical proof of equivalence between microscopic model and fluctuation model is still lacking. Still, the latter approach has very important property. In contrast to the microscopic model RG method could be used for analysing of its behavior and for obtaining quantitative predictions for critical exponents.

Now let's consider in detail theoretical approach to the problems of class b), whose study forms the major part of this dissertation work. The complete understanding of general fluid flow is one of the last prominent problems of classical physics, for which the solution is not known. Fluids can exhibit very different behavior from very simple such as laminar flow, which is very predictable, to very chaotic as is realized in turbulent motions. From our everyday experience we know that weather forecast could be done for no more than few days. This is caused by intrinsic instability of Navier-Stokes (NS) equations, that are believed to describe motion of viscous (non-relativistic) fluids [14].
For classification of various fluid states Reynolds number Re was introduced. It is defined as $\mathrm{Re} = VL/\nu$, where V is typical average flow velocity, L external scale (e.g. dimension of obstacle, that causes perturbation to the regular flow) and ν kinematic viscosity of the medium. It thus expresses a ratio between inertial and friction (dissipation) forces in given fluid. In the case of low values $\mathrm{Re} \ll 1$ regular (laminar) flow is observed. With increacing value of Re very different phenomena occur ranging from the periodical ones as Kármán vortices to very chaotic irregular motion for the limit of very high values of $\mathrm{Re} \gg 1$ (in practice value $\mathrm{Re} \geq 10^6$ is big enough) [15, 16]. This state of fluid is known as fully developed turbulence. At the first sight a very complicated problem turns out to be theoretically tracktable because of appearance of new symmetries (again kind of emergent symmetry) - statistical symmetries. Kolmogorov postulated hypothesis [17, 18] that could explain turbulence and also predict statistical and scaling properties of various correlation and structure functions. Kolmogorov theory can be considered as a kind of

theory for "ideal" turbulence in the sense it assumes infinite value for Reynolds number. These hypothesis are still not proved from the first principles - in this case from the Navier-Stokes equations.

The stochastic NS equations was proposed to justify Kolmogorov theory and has to be distinguished from the usual NS equations. Stochastic NS version can be considered as a kind of aforementioned fluctuation theory. The main difference is that for turbulence Hamiltonian (free-energy) functional is not known, resulting from the non-existence of equlibrium counterpart for turbulent regime. Stochastic NS equation neglects some effects as interactions influence of the boundaries or precise form of the system's geometry (e.g. information about grid structure), that are in the experiments responsible for creating turbulent instabilities. Also details of the mechanism of energy input is ignored and phenomenologically it is included by a proper choice of the stochastic force. The main goal of this theory is to justify Kolmogorov hypothesis. The general proof of the equivalence between Kolmogorov hypothesis and stochastic NS equations is still missing. Neverthless as various studies show it is possible to use stochastic Navier-Stokes equation also for modelling velocity fluctuations of different kinds.

Other interesting problem connected to the turbulence is an advection of some quantity [19]-[21] (temperature field, concentration field or tracer) by the turbulent field. Despite the practical importance of such problem it is also very interesting from the theoretical point of view. It is still not clear to what extent is turbulence intermittent [15], i.e. what is its fractal nature. On the other hand advection of passive scalar quantity by simpler models (e.g. Kraichnan model that will be described in detail later) than turbulence, exhibits very strong intermittent behavior.

The processes with non-conserved number of particles, such as irreversible chemical reactions [13], form another promiment members of class b). We would like to stress that irreversibility is very important feature of these models. The description of general reversible chemical reaction is sufficient within the means of kinetic rate equation. It is a consequence of ergodicity property, because all possible states can be reached. Thus in long time asymptotics $t \to \infty$ (that is always important for the practical issues) concetration of reactants and products attain their equilibrium values. However, in the case of irreversible chemical reactions the actual phase space is shrinking with growing time, because once particles react they are immediately throw out and they are not allowed to enter theoretical description. This brief overview explains the failure of traditional statistical physics for description of irreversible reactions. Other methods have to be used and one has always bear in mind the shortcomings and advantages of them. Also for them

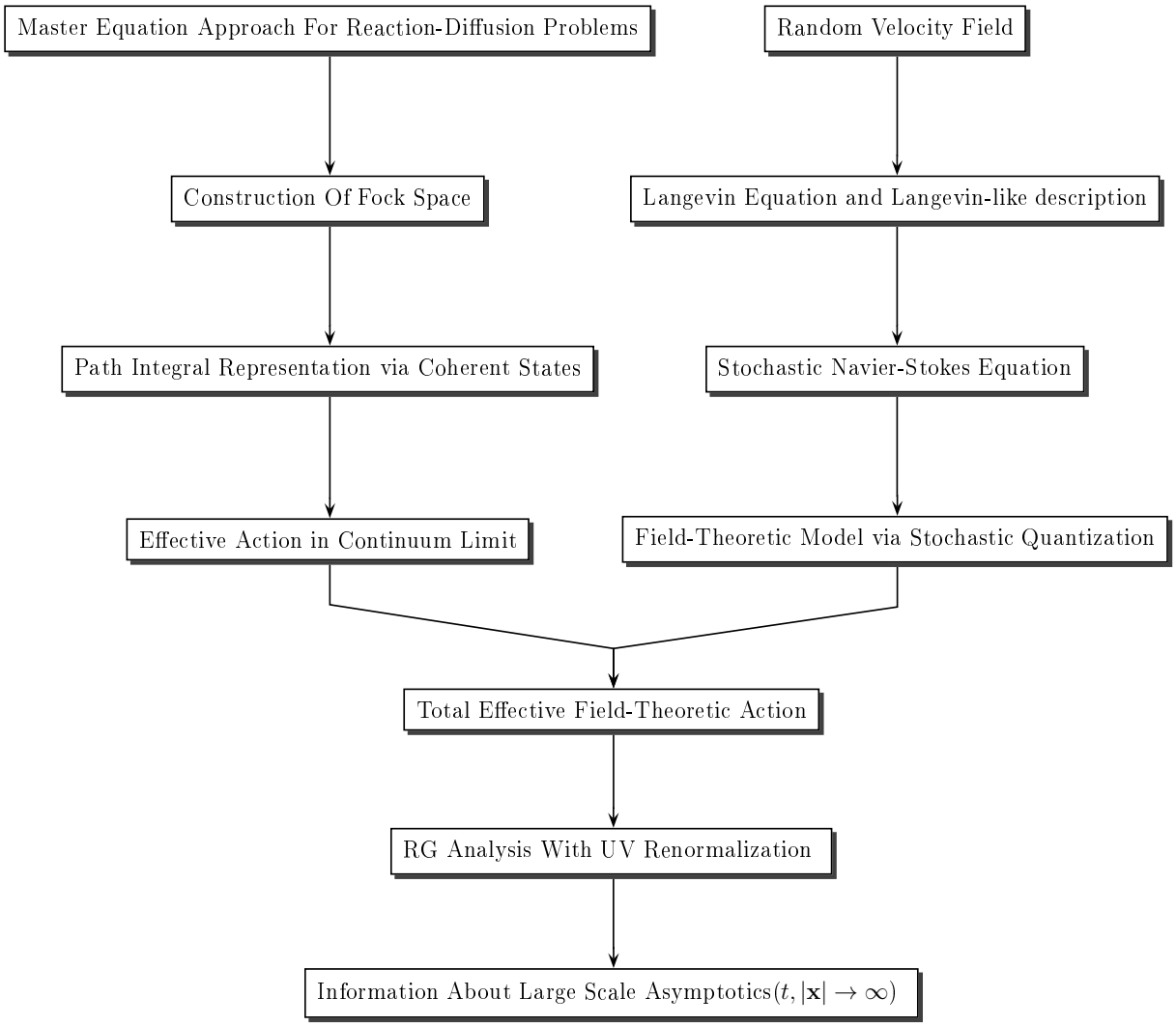


Figure 1: Overview of the approach for field-theoretic model of chemical processes advected by velocity fluctuations

Langevin-like description is possible, but as some studies show they contain unphysical features. For example for the single-species annihilation process $2A \to \varnothing$ it leads to the prediction of critical dimension to be $d_c = 6$ [22], which is in profound contradiction with the value $d_c = 2$, that is confirmed by the scaling analysis [23] and also by numerical calculations [24]. Another problem with such processes lies in the fact that their critical dimensions are usually very low ($d = 1$ or $d = 2$). If the movement of particles is due to diffusion, there appears profound density fluctuations (more precise anticorrelations between reacting particles) due to the re-entrancy property [25] of diffusive motion (continuos random walk). For this class of problems more rigorous treatment has to be used. So-called Doi approach [26] turns out to be a very efficient way how to deal with them. It

is based on the construction of mapping of classical system onto suitable "quantum" system. After identifying corresponding Fock space of quantum states, it is possible to recast it via coherent state into the path integral representation with some coarse-grained (continuum) effective action. And for analysing its asymptotic behavior one can use powerful RG technique.

Apart from very rare situations, the usual chemical reactions on earth take place in some environment, e.g. in liquid solution, pollutants in atmosphere, combustion processes in engines etc. Of course natural questions arise: what is the effect of the environment in which chemical reaction takes a place on it? Does it always increase (what we at the first sight expect) decaying rate? Or what are the actual parameters that controls time behavior of chemical process? The environment can be responsible for creation thermal fluctuations, atmosphere streams, random potentials or other perturbations and it is expected that they would have some additional effect on the reaction process. Given the properties of the environment the basic task is to quantify its influence on time behavior of the chemical process. For practical and also theoretical reasons usually large-scale (both space and time variable to be large) asymptotics is studied in non-equilibrium systems. The study of environment fluctuations on the time evolution for class of reaction-diffusion problem forms main part of the present work and for the better conception schematical sketch of our theoretical approach is depicted in Figure 1.

1.2 THESIS STATEMENT AND MAIN CONTRIBUTIONS

At present, there is no well-established theory for the out-of-equilibrium systems. In general, there is a tendency to study such systems separately and figure out some general principles. One of the most efficient approach is based on the use of perturbative renormalization group, which permits us to identify possible large-scale regimes and perturbation calculation of anomalous dimensions. Moreover there is also a demand for higher precision results (corresponds to the higher orders of the perturbation theory) for the anomalous corrections. The main reasons are

- Using the calculated coefficients in a given approximation there is an attempt to guess the structure of the whole asymptotic series and, eventually, possibly, to resum this series at least partially.

- To compare the obtained results in partial cases to the known exact results and thus to acknowledge them independently.

- To compare the results obtained within different perturbation schemes in order to develop better calculational techniques and algorithms.

- To compare the obtained results to the experimentally measured values or numerical simulations of studied quantities, thereby improving our understanding of the physical nature of the studied system.

The main task of this disseration is to study special class of reaction-diffusion problems in the presence of advecting velocity field. After construction of effective field-theoretic actions perturbative renormalization group technique will be used. In this dissertation we present the following contributions, which , in fact, are realizations of the main goals of the dissertation.

(a) We study the effect of turbulent fluctuations generated by stochastic Navier-Stokes equations on the annihilation problem $A + A \to \varnothing$. We present renormalization constants, RG functions and coordinates of fixed points to the second order approximation (Chapter 4), the effective Prandtl number and its dependence on the space dimensionality will be presented (Section 4.4).

(b) The asymptotic properties of the concentration field in the limit of large time values will be found together with leading one-loop correction to the rate equation (Section 4.5).

(c) We show how fluctuations described by Kraichnan with finite correlation time can affect the annihilation reaction, we found asymptotic solutions of the RG equations, coordinates of fixed points and their attraction regions will be estimated in the two-loop approximation (Section 5.4).

(d) We study the influence of compresibility introduced within the Kraichnan model on the annihilation reaction and we present fixed points of the RG equations with their attraction regions to the one-loop order (Section 6.4).

(e) We show how one can introduce sources and sinks into the field-theoretic description of general reaction scheme (Section 7.1) or annihilation process (Section 7.3).

(f) Effect of random fluctuations on the phase transition in the percolation problem is studied, the RG fixed points will be calculated and the attraction regions and their classification will be given (Chapter 8).

1.3 ORGANIZATION OF DISSERTATION

- In Chapter 2, we give a brief account on the stochastic Navier-Stokes equations, construction of corresponding field-theoretic model and how renormalization group approach can be used for analysing of its behavior.

- Chapter 3 is devoted to the detailed derivation of the Fock representation of reaction-diffusion problem and construction of effective field-theoretic action according to the Doi approach.

- In Chapter 4 annihilation process is studied in the presence of random velocity field governed by stochastic Navier-Stokes equations.

- In Chapter 5 Kraichnan-Obukhov model is introduced for the description of advecting field and its effect on the time evolution of annihilation process is studied.

- Chapter 6 presents field-theoretic study of the effect of compressibility on the behavior of annihilation process.

- Chapter 7 is devoted to the introduction of random sources and sinks into the field-theoretic model of reaction-diffusion processes.

- Chapter 8 presents results for the directed bond percolation model affected by the velocity fluctuations with long-range correlations.
- Sections A-D of Appendix contains detailed results for the singular parts of the Feynman graphs for all of presented models.
- Section E contains list of publications of the present author.

2 FIELD THEORETIC FORMULATION OF STOCHASTIC DYNAMICS

2.1 FIELD-THEORETIC DESCRIPTION OF STOCHASTIC NAVIER-STOKES EQUATIONS

The most suitable theoretical tool for the description of the developed turbulence is the statistical approach. A widely used method is based on the stochastic Navier-Stokes equation which governs the dynamics of the velocity fluctuations $\mathbf{v}(t,\mathbf{x})$

$$\partial_t \mathbf{v} + (\mathbf{v}.\boldsymbol{\nabla})\mathbf{v} - \nu_0 \nabla^2 \mathbf{v} + \nabla p = \mathbf{f}, \tag{2.1}$$

where ν_0 is the molecular kinematic viscosity and p stands for pressure fluctuations. For simplicity, we consider incompressible fluid with the solenoidal velocity $\boldsymbol{\nabla}.\boldsymbol{v} = 0$ and unit density of fluid ($\rho = 1$). Incompressibility condition permits elimination of the pressure from the Navier-Stokes equation and hence it is sufficient to consider only its transverse components

$$\partial_t \mathbf{v} + P(\mathbf{v}.\boldsymbol{\nabla})\mathbf{v} - \nu_0 \nabla^2 \mathbf{v} = \mathbf{f}. \tag{2.2}$$

Here, $\mathbf{f}$ represents an external random force per unit mass, which mimics an interaction between average smooth velocity components and fluctuations $\mathbf{v}$, $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2$ is the transverse projection operator (in momentum representation) where $k = |\mathbf{k}|$ is the norm of the wave vector $\mathbf{k}$. The large-scale random force $\mathbf{f}$ is assumed to be a Gaussian random variable with zero mean and the following correlation function

$$\langle f_m(\mathbf{x}_1,t_1) f_n(\mathbf{x}_2,t_2)\rangle = \delta(t_1 - t_2) \int \frac{d\mathbf{k}}{(2\pi)^d} P_{mn}(\mathbf{k}) d_f(k) \mathrm{e}^{\mathrm{i}\mathbf{k}.(\mathbf{x}_1 - \mathbf{x}_2)}, \tag{2.3}$$

where d is a dimension of space. Energy injection $d_f(k)$ is usually chosen in the form ([27])

$$d_f(k) = D_0 k^{4-d-2\epsilon} F(kL) \tag{2.4}$$

with the amplitude $D_0 = g_0 \nu_0^3$ and the scaling function $F(kL)$ with the unit asymptotic behavior in the range of large wave numbers $kL \gg 1$. The parameter g_0 plays the role of the bare coupling constant, $\epsilon \geq 0$ is a free parameter of the model, and L is an outer turbulent length. The term ν_0^3 is exctracted because of dimensional and calculational reasons.

Equation (2.1) from the mathematical point of view represents stochastic partial differential equation. The important fact about it is the appearance of only first time derivative ∂_t. Therefore it can be considered as a stochastic dynamics problem.

We recall the general statement for the equivalence between stochastic model and quantum field model [1]. The standard problem of stochastic dynamics could be described by the following general stochastic equation

$$\partial_t \varphi(x) = U(x,\varphi) + f(x), \quad \langle f(x) f(x')\rangle = D(x,x'), \tag{2.5}$$

where $x = (t, \mathbf{x})$ for convenience. Functional $U(x,\varphi)$ can be written as sum $L\varphi + n(\varphi)$, where $L\varphi$ represents linear part in field φ and $n(\varphi)$ contains non-linearities. Independence of U on time variable is very important.

Statement: An arbitrary stochastic problem (2.5) is equivalent to the field-theoretic model with double number of fields $\phi \equiv \{\varphi, \varphi'\}$ and with the action functional

$$S(\phi) \;=\; \frac{1}{2}\int\int dx dx' \varphi'(x) D(x,x') \varphi'(x') + \int dx\; \varphi'(x)[-\partial_t \varphi(x) + U(\varphi(x))]. \tag{2.6}$$

This statement is known as Martin-Siggia-Rosse mechanism (MSR) proposed by these authors in 1973 [28]. In the middle 70s it was re-formulated in the compact functional form by C. deDominicis, H. K. Janssen, R. Bausch and H. Wagner [29, 30] and later by Adzhemyan *et al.*[31].

In the language of the field-theoretic model the Green functions are defined as functional "averages" over the fields ϕ with weight $\exp(S)$, where S is action (2.6), thus they can be expressed as functional derivatives of the generating functional

$$G(A) = \int \mathscr{D}\phi \, \exp\left[S(\phi) + A\phi\right], \quad A\phi \equiv \int dx \left[\tilde{A}(x)\varphi(x) + A'(x)\varphi'(x)\right] \tag{2.7}$$

with respect to the formal sources $A \equiv (\tilde{A}, A')$ at $A = 0$. For example, the response function $\langle \varphi \varphi' \rangle$ is defined as

$$\langle \varphi(x)\varphi'(x')\rangle = \frac{\delta^2 G(A)}{\delta \tilde{A}(x) \delta A'(x')}\Big|_{A=0} = \int \mathscr{D}\phi \, \varphi(x)\varphi'(x') e^{S(\phi)}. \tag{2.8}$$

The normalization factor, which ensures the equality $G(0) = 1$, is included into the definition of the integral measure $\mathscr{D}\phi \equiv \mathscr{D}\varphi \mathscr{D}\varphi'$. In addition to the generating functional (2.7) physically important are connected Green functions given by the logarithm $W(A) = \ln G(A)$ and $1-$particle irreducible Green functions (in what follows 1PI) defined as func-

tional Legendre transform of $W(A)$

$$\Gamma(\Psi) = W(A) - A\Psi, \quad \Psi = \frac{\delta W(A)}{\delta A(x)}. \tag{2.9}$$

Finally, the action (2.5) can be rewritten as

$$S(\phi) = \frac{1}{2}\int\int dx dx' \varphi'(x) D(x,x') \varphi'(x') + \int dx \varphi'(x) \left[-\partial_t \varphi(x) + L\varphi(x) + n\left(\varphi\right)\right] \tag{2.10}$$

with the quadratic in fields part $S_0(\varphi, \varphi')$ and the interaction part $S_{int} = \varphi' n(\varphi)$.

Part S_0 can be rewritten in the symmetric form

$$S_0(\varphi,\varphi') \equiv -\frac{1}{2}\phi K \phi \equiv -\frac{1}{2}\begin{pmatrix} \varphi \\ \\ \varphi' \end{pmatrix}\begin{pmatrix} 0 & (\partial_t - L)^T \\ \\ \partial_t - L & -D \end{pmatrix}\begin{pmatrix} \varphi \\ \\ \varphi' \end{pmatrix}, \tag{2.11}$$

with the symmetric matrix K, where T represents the operation $K^T(x,x') = K(x',x)$. The inverse matrix $\Delta = K^{-1}$ defines a set of bare propagators

$$\begin{aligned} &\Delta_{ik}(x,x') = \langle \phi_i(x), \phi_k(x') \rangle_0 , \\ &\Delta_{12} = \Delta_{21}^T = (\partial_t - L)^{-1}, \quad \Delta_{11} = \Delta_{12} D \Delta_{21}, \quad \Delta_{22} = 0, \end{aligned} \tag{2.12}$$

where $\phi_1 \equiv \varphi$, $\phi_2 \equiv \varphi'$. The propagator Δ_{12} is retarded, therefore $\Delta_{21} = \Delta_{12}^T$ is advanced. The symmetric propagator $\Delta_{11} = \Delta_{11}^T$ contains both (retarded and advanced) contributions. The interaction part generates vertices with one field φ' and two or more fields φ, which are dictated by the concrete form of the nonlinear terms in the action of our model. The aforementioned representation permits the construction of standard Feynman graphs for Green functions. The lines (propagators) are derived from the quadratic (free) part S_0 and vertices are defined by S_{int} by means of the Wick theorem (see, e.g.[32]), that can be compactly written in the exponential form for the functional (2.7) as

$$G(A) = \exp\Big(\frac{1}{2}\frac{\delta}{\delta\phi}\Delta\frac{\delta}{\delta\phi}\Big)\exp\Big(S_{int}(\phi) + A\phi\Big)\Big|_{\phi=0}, \tag{2.13}$$

where Δ is the matrix of propagators (2.12) and

$$\frac{\delta}{\delta\phi}\Delta\frac{\delta}{\delta\phi} \equiv \int dx dx' \frac{\delta}{\delta\phi_i(x)}\Delta_{ik}(x,x')\frac{\delta}{\delta\phi_k(x')} \tag{2.14}$$

is a shorthand notation for universal differential operation and summation over repeated indices is implied. Expanding both exponents in (2.13) all Green functions as formal infinite series in Feynman graphs are obtained.

Let us demonstrate how these general rules work in the theory of developed turbulence. According to the aforementioned MSR mechanism the stochastic model described by Equation (2.5) is equivalent to the field-theoretic model with De Dominicis-Janssen-Vasiliev action (2.10), which in this case is of the following form

$$\begin{aligned} S(\mathbf{v},\tilde{\mathbf{v}}) &= \frac{g_0\nu_0^3}{2}\int\int dxdx'\,\tilde{v}_i(x)\tilde{D}_{ij}(x,x')\tilde{v}_j(x') + \\ &+ \int dx\,\tilde{\mathbf{v}}(x)\cdot\left[-\partial_t\mathbf{v}(x)+\nu_0\nabla^2\mathbf{v}(x)-(\mathbf{v}(x)\cdot\boldsymbol{\nabla})\mathbf{v}(x)\right], \end{aligned} \tag{2.15}$$

where the auxiliary vector field $\tilde{\mathbf{v}}$ is solenoidal ($\boldsymbol{\nabla}.\tilde{\mathbf{v}}=0$) like the velocity field $\mathbf{v}$, and ν_0 is a bare (molecular) viscosity coefficient. To distinguish it from the renormalized (turbulent) viscosity ν, which appears in the process of the renormalization procedure (see below) we mark it and other analogous (bare) parameters by the subscript "zero". We stress that this notation will be used in whole work. The noise D_{ij} (see Equation (2.15)) we rewrite in form $g_0\nu_0^3\tilde{D}_{ij}$ which is more convenient for the further analysis. In the case of an incompressible fluid the contribution of pressure into the action (2.15) vanishes due to the condition $\boldsymbol{\nabla}.\tilde{\mathbf{v}}=0$. By means of the general operation (2.13) one obtains Feynman

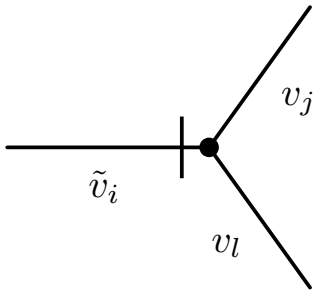
$$v_i \longrightarrow v_j = \langle v_i v_j\rangle_0 \equiv \Delta_{ij}^{vv}(\omega_k,\mathbf{k})$$

$$v_i \longrightarrow \tilde{v}_j = \langle v_i \tilde{v}_j\rangle_0 \equiv \Delta_{ij}^{v\tilde{v}}(\omega_k,\mathbf{k})$$

$$\tilde{v}_i \longrightarrow \tilde{v}_j = \langle \tilde{v}_i \tilde{v}_j\rangle_0 \equiv \Delta^{\tilde{v}\tilde{v}}(\omega_k,\mathbf{k})$$

Figure 2: Propagators of the model

rules for the propagators (lines) Δ and vertices V, which are depicted in Figure 2 and Figure 3. The explicit form of the propagators can be obtained from the quadratic part of action (2.15) and in the wave-vector-momentum representation they have the following



$$\equiv V_{ijl} = i(k_j\delta_{il} + k_l\delta_{ij})$$

Figure 3: Interaction vertex responsible for the nonlinear interactions between velocity fluctuations)

form

$$\Delta_{ij}^{vv}(\boldsymbol{k},\omega_k) = \frac{P_{ij}(\mathbf{k})D(k)}{(i\omega_k+\nu_0k^2)(-i\omega_k+\nu_0k^2)},$$

$$\Delta_{ij}^{v\tilde{v}}(\boldsymbol{k},\omega_k) = \frac{P_{ij}(\mathbf{k})}{-i\omega_k+\nu_0k^2}, \tag{2.16}$$

$$\Delta_{ij}^{\tilde{v}v}(\boldsymbol{k},\omega_k) = \frac{P_{ij}(\mathbf{k})}{i\omega_k+\nu_0k^2}, \tag{2.17}$$

$$\Delta_{ij}^{\tilde{v}\tilde{v}}(\boldsymbol{k},\omega_k) = 0, \tag{2.18}$$

and in the time-momentum representation

$$\Delta_{ij}^{vv}(\mathbf{k},t'-t) = \frac{P_{ij}(\mathbf{k})D(k)}{2\nu_0k^2}\exp\Big(-\nu_0k^2\left|t'-t\right|\Big), \tag{2.19}$$

$$\Delta_{ij}^{v\tilde{v}}(\mathbf{k},t'-t) = \theta\left(t'-t\right)P_{ij}(\mathbf{k})\exp\Big(-\nu_0k^2\left(t'-t\right)\Big), \tag{2.20}$$

$$\Delta_{ij}^{\tilde{v}v}(\mathbf{k},t'-t) = \theta\left(t-t'\right)P_{ij}(\mathbf{k})\exp\Big(-\nu_0k^2\left(t'-t\right)\Big), \tag{2.21}$$

$$\Delta_{ij}^{\tilde{v}v}(\mathbf{k},t'-t) = 0, \tag{2.22}$$

where $P_{ij}(\mathbf{k}) = \delta_{ij} - k_ik_j/k^2$ is the transverse projector due to incompressibility. Here the step function θ reflects an important physical feature of the propagator $\Delta^{v\tilde{v}}$ namely, its retardation, because actually it is the lowest order approximation of the response function $\langle v\tilde{v}\rangle$ of the original model (2.1)–(2.4). The propagator Δ^{vv} represents the leading approximation of the pair correlation function of the velocity field $W_{ij} = \langle v_i\, v_j\rangle$, which in the wave-vector representation is proportional to the kinetic energy spectrum $E(k)$. As it is well known $E(k)$ plays very important role in the equation of energy balance, which describes the cascade of the kinetic energy from the largest scales to the smallest ones,

where it dissipates [15]. The vertex factor

$$V_m(x_1, x_2, \ldots, x_m; \Phi) = \frac{\delta^m V(\Phi)}{\delta\Phi(x_1)\delta\Phi(x_2)\ldots\delta\Phi(x_m)} \tag{2.23}$$

is associated to each interaction vertex of Feynman graph. Here, Φ could be any member from the set of all fields $\{\tilde{v}, v\}$. The interaction vertex in (2.15) may be rewritten in a technically more convenient form

$$-\int dtd\mathbf{x}\, \tilde{\mathbf{v}}(\mathbf{v}.\nabla)\mathbf{v} = -\int dtd\mathbf{x}\, \tilde{v}_i v_k \partial_k v_i = \int dtd\mathbf{x}\, (\partial_k \tilde{v}_i) v_k v_i, \tag{2.24}$$

where the incompressibility condition $\partial_i v_i = 0$ and partial integration method have been used. We assume that the velocity fields fall off rapidly enough for $|\mathbf{x}| \to \infty$. Rewriting this functional in the symmetric form $v_i V_{ijl} v_j v_l/2$, it is easy to find the explicit form for the corresponding vertex factor in the momentum space

$$V_{ijl} = i(k_j \delta_{il} + k_l \delta_{ij}). \tag{2.25}$$

Here, the momentum $\mathbf{k}$ is flowing into the vertex through the field $\tilde{v}$ and is denoted by slash in Figure 3.

2.2 RENORMALIZATION PROCEDURE AND RENORMALIZATION GROUP

Usually, the Feynman graphs of Green functions, which are the graphical representation of some integrals, contain divergences in the range of large and small scales (wave vectors.) Therefore it is necessary to find an effective procedure to eliminate these divergences step by step in each order of concrete perturbation scheme. Below we will demonstrate renormalization methods in the framework of the stochastic model of developed turbulence (2.1) - (2.4).

The method of renormalization group (RG) has been proposed in the framework of the quantum field theory in the 50s of the previous century [4]-[9]. From the practical point of view RG method represents an effective way to determine non-trivial asymptotic behavior of Green functions in the range of large (ultraviolet) or small (infrared) wave vectors (scales). The asymptotic behavior is non-trivial if in a given order of a perturbative calculation the divergences in a certain range of wave vectors appear (e.g. so called large logarithms) which compensate the smallness of coupling constant g. In such a case one needs to sum all terms of perturbation series. This summation can be carried out by

means of RG approach. Technically, one obtains linear partial differential RG equations for the Green functions. The coefficient functions (RG-functions) in the differential operator (see below) are calculated at a given order of the perturbation scheme. However, the solution of the RG equation represents the sum of the infinite series. For example, if the RG-functions are calculated at the lowest non-trivial order of the perturbation theory and after corresponding RG-equation is solved, obtained result is a sum of leading logarithms of all the perturbation series. Moreover if the RG-functions will be calculated with an improved precision the solution of the RG equation will include corrections to the leading logarithms.

A simple criteria how to determine the true asymptotic range exist in the framework of RG. One of the RG-functions is the β-function, which is a coefficient at the operation ∂_g in the RG equation. The β-function is calculated perturbatively as infinite series of powers of the coupling constant g and for relativistic models has form: $\beta(g) = \beta_2 g^2 + \beta_3 g^3 + ...$ Non-trivial asymtotic behavior is governed by *RG fixed points* g_*, which are roots of β-functions (solutions of equation $\beta(g) = 0$). A fixed point can be infra-red (IR) or ultra-violet stable depending on behavior of the β-function in the vicinity g_*. If the coeficient $\beta_2 > 0$ then g_* is an IR stable fixed point. In the opposite case it is UV stable fixed point. In the time when the RG technique appeared no physical models with non-trivial UV asymptotic behavior were known. Moreover, as non-trivial IR behavior is possible only for massless models, which also were not known in this time, RG method remained unused up to the seventies of the previous century. In this time two new areas appeared, where the RG technique could be applied:

(1) In the relativistic high energy physics it was connected with discovery of so called *UV asymptotic freedom* in the Yang-Mills theories containing non-abelian gauge fields [33, 34], where it was shown that the coefficient β_2 is negative so the fixed point governs the UV asymptotic behavior. Recently (2004) D. J. Gross, F. Wilczek and H. D. Politzer have been awarded the Nobel prise for this discovery.

(2) The theory of critical phenomena [35]-[42] represents second area, where the study of the nontrivial IR asymptotic behavior by the RG method is interesting because at the critical point (point of phase transition) the effective universal models correspond to the massless fields. Let us remind that usually the critical behavior of a system under consideration is understood as its behavior near the critical point of a phase transition of the second order , where the scaling with nontrivial exponents appears. As an example, we can present the known critical exponent β characterizing the behavior

of the magnetization of a ferromagnet in zero external field near critical temperature T_c

$$M \sim (T_c - T)^{\beta}, \quad T < T_c. \tag{2.26}$$

Calculation of the critical exponents, explanation of their universality and finding relations between them is the subject of study of the theory of critical behavior. RG represents a powerful tool for the solution of mentioned problems. First, the RG approach was applied by K. G. Wilson [42] (Wilson renormalization group) in critical statics and afterwards used also to the study critical dynamics [43, 44].

In 1977 D. Forster, D. R. Nelson and M. J. Stephen applied the RG method to calculate the correlations of velocity field [45] governed by stochastic Navier-Stokes equation with external random forcing. Later it was shown by C. de Dominicis and P. C. Martin [27] that in the range of small wave numbers the correlations of the velocity field manifest a scaling behavior with the celebrated Kolmogorov exponents. The basic idea of application of RG in the theory of developed turbulence consists in elimination of the direct influence of the modes with high frequencies and wave numbers on observed quantities. Effectively their influence is included to some effective variables, e.g. to the turbulent viscosity.

Numerous versions of RG methods exist. Constricted all they are equivalent but technically can be quite different. The most formalized is RG developed in the framework of the quantum field theory (field-theoretic RG), which has been used in papers [30, 31].

The field theoretic RG is based on non-trivial techniques of ultra-violet (UV) renormalization. The basic procedure consists of calculation of the RG-functions in the framework of a prescribed scheme of regularization [2]. To find and analyze all possible UV divergences in concrete field-theoretic models a counting of canonical scaling dimensions of fields and parameters of the model is used. Let us briefly remind the essence of such a power counting which is closely connected with the existence of a scale invariance in the model.

2.3 EXTENDED HOMOGENEITY

This conception is introduced for the formulation of the hypothesis of similarity (critical scaling, scale invariance) and it is useful for classification of all UV divergencies in field-theoretical models: critical statics and dynamics, developed turbulence and so on [2, 1]. We give its definition and basic properties.

A function $W(e)$ which depends on variables $e \equiv \{e_1 ... e_2\}$ is termed extended homogeneous (or simply speaking dimensional) if for a set of numbers Δ and arbitrary $\lambda > 0$ the following equation is valid

$$W(\lambda^{\Delta_1} e_1, ..., \lambda^{\Delta_n} e_n) = \lambda^{\Delta_W} W(e_1, ..., e_n), \tag{2.27}$$

or shortly

$$W(e_\lambda) = \lambda^{\Delta_W} W(e), \tag{2.28}$$

where $e_\lambda = \lambda^{\Delta_i} e_i$ and $i = 1, ..., n$. The parameters $\Delta_i \equiv \Delta_{e_i} \equiv \Delta[e_i]$ are (canonical or critical or anomalous) scaling dimensions (exponents) of corresponding variables e_i and $\Delta_W \equiv \Delta[W]$ represents dimension (exponent) of function W. If $\Delta_W = 0$ then the function W is scale invariant (dimensionless). A dimensional function $W(e)$ depending on one variable e is proportional to the power $|e|^\beta$ with the exponent $\beta = \Delta_W/\Delta_e$. A function W depending on n variables can be expressed in the form of product of a power function and *scaling function* f which is a function of $n-1$ dimensionless combinations of its arguments, e.g.

$$W(e_1, \ldots, e_n) = |e_n|^\beta f\left(\frac{e_1}{|e_n|^{\beta_1}}, \ldots, \frac{e_{n-1}}{|e_n|^{\beta_{n-1}}}\right), \tag{2.29}$$

where $\beta = \Delta_W/\Delta_n$ and $\beta_i = \Delta_i/\Delta_n$ $i = 1, 2, \ldots n-1$. If one differentiates the equation (2.28) with respect to λ and afterwards puts $\lambda = 1$, then will obtain differential equation, which represents another equivalent formulation of the extended homogenity (2.28)

$$\left[\sum_e \Delta_e e \partial_e - \Delta_W\right] W(e) = 0. \tag{2.30}$$

The substitution $\lambda \to \lambda^a$ in the equation (2.28) is equivalent to multiplying of all exponents Δ by parameter a, therefore the exponent of one of the variables can be fixed. Usually the dimension of the wave number k is selected to be the unity (or, equivalently, dimension of the space coordinate is equal to -1)

$$\Delta[k] = 1. \tag{2.31}$$

This definition is standard and we use it everywhere. We note that in dynamical models (2.5) also the dimension of frequency can be fixed (see below).

2.4 UV RENORMALIZATION AND RG EQUATIONS

Let us expound brief information necessary on the quantum-field theory of renormalization and RG technique; a detailed account can be found in monographs [2] and [46].

We will consider models whose diagrams are calculated without UV-cut-off Λ and UV-divergences are observed as poles in a certain dimensionless "parameter of deviation from logarithmic theory ϵ". Historically, this term appeared in connection with infinite summation of main logarithms (see aforementioned discussion)), which is necessary to make in the case when the bare coupling constant g_0 (or constants) is canonically dimensionless ($\Delta_g \equiv d_g = 0$). The procedure of multiplicative renormalization removing UV-divergences (in the given case, poles in a parameter ϵ) consists in the following: the original action $S(\phi, e_0)$ is declared to be unrenormalized; its parameters e_0 (letter e_0 stands for the whole set of parameters) are the bare parameters, and they are considered to be some functions of the new renormalized parameters e, whereas a new renormalized action is assumed to be the functional $S_R(\phi) = S(\phi Z_\phi)$ with certain (also to be determined) renormalization constants of fields Z_ϕ (one per each independent component of the field). In unrenormalized full Green functions $G_n = \langle \Phi \dots \Phi \rangle$ the functional averaging $\langle \dots \rangle$ is performed with the "weight" $\exp S(\phi)$; while in renormalized functions , G_n^R with the "weight" $\exp S_R(\phi)$. The connection between the functionals $S(\phi)$ and $S_R(\phi)$ leads to the relation between the corresponding Green functions $G_n^R = Z_\Phi^{-n} G_n$, where by definition $G_n = G_n(e_0, \epsilon \dots)$(ellipsis denotes other arguments like coordinates or wave numbers), and, by convention, the quantities G_n^R and Z_Φ are expressed in terms of the parameters e. The correspondence $e_0 \leftrightarrow e$ within perturbation theory is assumed to be one-to-one, therefore either of the sets e_0, e can be taken as the independent variables.

Very often it is convenient to deal not with the full Green functions G_n, but either with their connected parts W_n (their generating functional being $W(A) = \ln G(A)$) or with 1-irreducible functions Γ_n, which generating functional is defined by the functional Legendre transform (2.9). These unrenormalized and renormalized Green fuctions satisfy relation

$$W_n^R(e, \epsilon, \dots) = Z_\varphi^{-n}(e, \epsilon) W_n(e_0(e, \epsilon), \epsilon, \dots), \tag{2.32}$$

$$\Gamma_n^R(e, \epsilon, \dots) = Z_\varphi^{n}(e, \epsilon) \Gamma_n(e_0(e, \epsilon), \epsilon, \dots), \tag{2.33}$$

where the functions $e_0(e, \epsilon)$, $Z_\varphi^n(e, \epsilon)$ can be chosen arbitrarily, which implies an arbitrary choice of normalization of the fields and parameters e at given e_0. The basic statement

of the theory of renormalization is that for the multiplicatively renormalizable models these functions can be chosen to provide UV-finiteness of Green functions as $\epsilon \to 0$. With this choice, all UV-divergences (poles in ϵ) contained in the functions $e_0(e,\epsilon)$, $Z_\varphi^n(e,\epsilon)$ are absent in renormalized Green functions $W_n^R(e,\epsilon)$. We note that the UV-finiteness in this sense of any one set of Green functions (full, connected, 1-irreducible) automatically leads to the UV-finiteness of any other. The RG equations are written for the renormalized functions W_n^R which differ from the original unrenormalized functions W_n only by normalization, and therefore, can be used equally well to analyze the critical scaling. Let us demonstrate an elementary derivation of the RG equations [1]. The requirement of elimination of divergences does not uniquely determine the functions $e_0(e,\epsilon)$ and $Z_\varphi(e,\epsilon)$. An arbitrariness remains which allows to introduce in these functions (and via them also into W_n^R) an additional dimensional parameter - scale setting parameter (renormalization mass) μ

$$W_n^R(e,\mu,\epsilon,\dots) = Z_\varphi^{-n}(e,\mu,\epsilon) W_n(e_0(e,\mu,\epsilon),\epsilon,\dots). \tag{2.34}$$

A change of μ at fixed e_0 leads to a change of e, Z_φ and W_R for unchanged $W_n(e_0,\epsilon,\dots)$. We use $\tilde{\mathscr{D}}_\mu$ to denote the differential operator $\mu\partial_\mu$ for fixed e_0 and operate on both sides of the equation $Z_\varphi^n W_n^R = W_n$ with it. This gives the basic RG differential equation

$$\left[\mu\partial_\mu + \sum_e \tilde{\mathscr{D}}_\mu e \partial_e + n\gamma_\varphi\right] W_n^R(e,\mu,\epsilon,\dots) = 0, \quad \gamma_\varphi \equiv \tilde{\mathscr{D}}_\mu log Z_\varphi, \tag{2.35}$$

where the operator $\tilde{\mathscr{D}}_\mu$ is expressed in the variables μ, e. The coefficients $\tilde{\mathscr{D}}_\mu e$ and γ_φ are called the RG functions and are calculated in terms of various renormalization constants Z. All the RG-functions are UV-finite, i.e. have no poles in ϵ, which is a consequence of the functions W_n^R being UV-finite in (2.35).

The general theory of renormalization [46] distinguishes unrenormalized S, renormalized S_R, and base S_B actions; the last is obtained from S by replacement of all the bare parameters e_0 by their renormalized counterparts e. The UV-divergences are removed by adding to the base action S_B all necessary counterterms ΔS which are determined by the known rules (see below). If the renormalized action thus obtained $S_R = S_B + \Delta S$ can be reproduced by the above procedure of redefinition of fields and parameters in the original unrenormalized action S, the model is multiplicatively renormalizable. Therefore, the first step in the RG analysis of any model is to explicitly determine all counter-terms required for the removal of UV-divergences and to verify its multiplicative renormalizability.

The form of the required counterterms is determined by the analysis of canonical dimensions of the 1-irreducible Green functions of model with the action S_B, which satisfy the equation of extended homogenity (2.27) (or, equivalently, (2.30)) with definite canonical exponents $\Delta_e \equiv d_e, d_\mu = 1, d_\phi$ of parameters e_0, e, μ and fields ϕ, respectively. Dynamical models (2.5), in contrast to static ones, are two-scale models, i.e. two independent canonical wave-number (momentum) d_k^Q and frequency d_ω^Q dimensions can be assigned to every quantity Q (fields and parameters in the action). They are easily determined from the natural normalization conditions $d_k^k = -d_x^k = 1$ a $d_\omega^\omega = -d_t^\omega = 1$ and from the requirement that every term T of the actions S, S_B is dimensionless ($d_k^T = d_\omega^T = 0$). After that summarized full canonical dimension d^Q can be determined by means of d_k^Q and d_ω^Q. In general, determination of this quantity depends on the actual model. For the model of stochastic developed turbulence (2.1) - (2.4) it has to be set $d_Q = d_k^Q + 2d_\omega^Q$ since the action of corresponding field theory contains the combination $\partial_t - \nu_0 \triangle$ ($-i\omega + \nu_0 k^2$ in wave-number-frequency representation). It implies $\omega \sim k^2$. Of course, the existence of the two aforementioned wave-number and frequency scale invariance can be expressed by means of two differential equations similar to equation (2.30) with corresponding exponents d_k^e, d_ω^e of parameters e.

In the scheme of renormalization of dynamical models (2.5) the full dimension d^Q plays the same role as the conventional (momentum) dimension does in static problems. Canonical dimensions of an arbitrary 1-irreducible Green function Γ with n_ϕ (multiple index) fields φ', φ for a $d+1$-dimensional problem are given by the relations

$$d_\Gamma^k = d - \sum_\phi n_\phi d_\phi^k, \;\; d_\Gamma^\omega = d - \sum_\phi n_\phi d_\phi^\omega, \;\; d_\Gamma = d + 2 - \sum_\phi n_\phi d_\phi \tag{2.36}$$

with summation over all the fields ϕ entering into given function Γ. In a logarithmic theory, which corresponds to $\epsilon = 0$ when bare coupling constant(s) g of concrete model is (are) dimensionless ($d^g = d_k^g = d_\omega^g = 0$), full canonical dimension of Γ_n is equal to a formal index of UV-divergence δ. The UV-divergences which must be removed by suitable counterterms are allowable only in those functions Γ_n for which index δ is nonnegative and integer [1]. All counterterms are polynomial functions of wave vector $\mathbf{k}$ and frequency ω.

In models considered in the present dissertation the analysis of divergences should be based on the following auxiliary considerations:

(a) For any dynamic model (2.5) all 1-irreducible Green functions containing only the original fields φ are proportional to the closed circles of step functions hence they

vanish, and thus do not generate counterterms.

(b) If for some reason several external momenta or frequencies occur as an overall factor in all the graphs of a particular Green function, the real degree of divergence δ' is less than $\delta \equiv d_\Gamma|_{\varepsilon=0}$ by the corresponding number of units.

(c) Sometimes the divergences formally allowed dimensionally are absent due to symmetry requirements, for instance, the Galilean invariance of model (2.1).

(d) Nonlocal terms of the model are not renormalized.

These general considerations and formula (2.36) permit us to determine all superficially divergent functions and to explicitly obtain the corresponding counter-terms for any concrete dynamic model.

2.5 RG APPROACH TO FULLY DEVELOPED TURBULENCE

In this subsection we will show how the general rules of renormalization can be applied to the stochastic model of developed turbulence. It is useful from a pedagogical point of view because all steps demonstrated below can be relatively easily generalized to more complicated and sophisticated models associated with stochastic turbulence in the framework of which besides basic quantities like correlation functions one is able to calculate correlation functions of composite fields, i.e. composite operators. The later are physically important because they are associated with experimentally measurable quantities standing in the energy balance equation e.g. energy dissipation rate, energy transfer function and so on [32, 36, 47, 48]. First such RG analysis have been carried out in [45, 31].
Let us start with the renormalization. The base action corresponding to the unrenormalized action (2.15) is of the form

$$S_B(\mathbf{v},\tilde{\mathbf{v}}) = \frac{\mu^{2\epsilon} g \nu^3}{2} \tilde{\mathbf{v}} D \tilde{\mathbf{v}} + \tilde{\mathbf{v}} \cdot \left[-\partial_t \mathbf{v} + \nu \nabla^2 \mathbf{v} - (\mathbf{v}.\boldsymbol{\nabla})\mathbf{v} \right], \tag{2.37}$$

where summations over indices of the transversal vector fields $\mathbf{v}, \tilde{\mathbf{v}}$ and integrations over time and coordinates (or frequency and wave vectors) are implied. Here μ is the renormalization mass (scale setting parameter). In this case $e_0 = \{g_0, \nu_0\}$ are the bare parameters, and $e = \{g, \nu\}$ are the corresponding renormalized parameters.

The canonical dimensions of fields and parameters of the model for an arbitrary space dimension d are presented in Table 1. Using these dimensions we find from (2.36) the

F	$\mathbf{v}$	$\tilde{\mathbf{v}}$	L,μ	ν,ν_0	g_0	g
d_F^p	-1	$d+1$	-1	-2	2ϵ	0
d_F^ω	1	-1	0	1	0	0
d_F	1	$d-1$	-1	0	2ϵ	0

Table 1: Canonical dimensions for the fields and parameters of the field-theoretic model for stochastic Navier-Stokes equations

formal UV-divergence index $\delta = d_\Gamma$

$$\delta = d + 2 - n_v - (d-1)n_{\tilde{v}}, \tag{2.38}$$

where n_v and $n_{\tilde{v}}$ are numbers of the corresponding fields entering the one-irreducible functions Γ_n. From the explicit form of the vertex (2.25) we see that the derivative ∇ may always be transferred to the external field $\tilde{\mathbf{v}}$ and therefore the real UV-index actually is

$$\delta' = \delta - n_{\tilde{v}} = d + 2 - n_v - dn_{\tilde{v}}. \tag{2.39}$$

From the forms of δ and δ' it results that for $d > 2$ the UV divergencies are present only in 1-irreducible functions $\langle \tilde{v}v \rangle$ ($\delta = 2$, $\delta' = 1$) and $\langle \tilde{v}vv \rangle$ ($\delta = 1$, $\delta' = 0$), whereas the corresponding counterterms should contain the operator $\boldsymbol{\nabla}$. Therefore, the first function generates only the counterterm $\tilde{v}\boldsymbol{\nabla}^2 v$, but does not contribute to the term $\tilde{v}\partial_t v$. The function $\langle \tilde{v}vv \rangle$ generates counterterm containing only one gradient $\boldsymbol{\nabla}$, which always can be reduced to the form $\tilde{v}(\boldsymbol{\nabla} v)v$ due to transversality of fields considered. However, this counterterm is actually forbidden due to Galilean invariance of model (2.1)-(2.4) (and of course action (2.15)) Therefore, if there is no counter-term $\sim v'\partial_t v$, there is no counter-term $\sim v'(v\nabla)v$ as well. In a particular case $d = 2$, there appears a new superficial divergence in the function $\langle v'v' \rangle$ ($\delta = 2, \delta' = 0$), which generates the counter-term $\sim v'\partial^2 v'$. The two-dimensional problem has been treated in refs. [49] and in more sophisticated problems in papers [50, 51, 52]. In the last reference a special method to fuse solutions obtained for both $d = 2$ $d = 3$ has been elaborated in the framework of a special double expansion scheme. For $d > 2$ only one counter-term $\sim \tilde{v}\boldsymbol{\nabla}^2 v$ is required. Its addition to (2.37) leads to the renormalized action

$$S_R(\mathbf{v},\tilde{\mathbf{v}}) = \frac{\mu^{2\epsilon} g \nu^3}{2}\tilde{\mathbf{v}} D \tilde{\mathbf{v}} + \tilde{\mathbf{v}} \cdot \left[-\partial_t \mathbf{v} + Z_\nu \nu \nabla^2 \mathbf{v} - (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} \right], \tag{2.40}$$

where Z_ν is the renormalization constant. It is fully dimensionless and thus can depend on the only dimensionless renormalized parameter g (dependence on ϵ and d is always implied).

The explicit form of Z_ν depends on the choice of subtraction scheme. The purpose of a counterterm is to cancel out the poles in ϵ in diagrams, therefore the coefficient of a counterterm should contain such poles. However, its finite part can be chosen arbitrarily, and its fixation is just the choice of subtraction scheme. The most convenient for practical computation is the minimal subtraction scheme (MS) [53] in which counterterms contain only poles in ϵ and no finite contributions. Physical results do not depend on the choice of a particular scheme, therefore such a choice is a question of convenience, and we will always utilize the widely accepted MS scheme. In this scheme, only ϵ-poles are subtracted from the diagrams and the renormalization constants Z are of the following form

$$Z = 1 + \sum_{k=1}^{\infty} a_k(g)\epsilon^{-k} = 1 + \sum_{n=1}^{\infty} g^n \sum_{k=1}^{n} a_{nk}\epsilon^{-k}. \tag{2.41}$$

The coefficients a_{nk} in model (2.40) can depend only on the space dimension d; the absence of ϵ in residues a_{nk} is a specific feature of the MS scheme.

The renormalized action (2.40) is obtained from the nonrenormalized one (2.15) by the following renormalization of parameters

$$\nu_0 = \nu Z_\nu, \quad g_0 = g\mu^{2\epsilon} Z_g, \quad Z_g = Z_\nu^{-3}, \quad Z_v = Z_{\tilde{v}} = 1. \tag{2.42}$$

The model is multiplicatively renormalizable and a standard scheme of derivation of RG equations of type (2.35) is applicable to it; also, in this case $\gamma_\varphi = \gamma_v = \gamma_{\tilde{v}} = 0$ because there is no field renormalization. Hence there is only one independent renormalization constant Z_ν, whereas Z_g is expressed in terms of Z_ν because there is no renormalization of nonlocal contribution of the random force correlator in (2.4). Generally, this correlator depends on outer turbulent scale L , but it is not renormalized hence to the all formulas of renormalization the simple relation is added

$$L_0 = Z_L L\,,\ Z_L = 1\,. \tag{2.43}$$

All renormalization constants are calculated directly from diagrams generated by the base action (2.37). In the general case, using the renormalization constant Z_F of any quantity F (field or parameter), one can determine the corresponding RG-function γ_F

– *the anomalous dimension (exponent, index)* of F, and the β-functions of any charge g (fully dimensionless expansion parameter) on which the constants Z_F and anomalous exponents can depend

$$\beta_g = \mu\partial_\mu g, \quad \gamma_F = \mu\partial_\mu \ln Z_F = \beta_g \partial_g \ln Z_F. \tag{2.44}$$

Using expression (2.42) one obtains

$$\gamma_g = -3\gamma_\nu, \quad \beta_g = -g(2\epsilon + \gamma_g) = g(-2\epsilon + 3\gamma_\nu). \tag{2.45}$$

The RG-operator in the general RG-equation (2.35) takes the form

$$D_{RG} = \mu\partial_\mu + \beta_g\partial_g - \gamma_\nu(g)\nu\partial_\nu. \tag{2.46}$$

One-loop calculations of the UV-divergent part of the 1-irreducible function corresponding to the response function $\langle v\tilde{v}\rangle$ of model (2.15) gives [1]

$$Z_\nu = 1 - \frac{cg}{2\epsilon} + O(g^2), \quad c = \frac{C_d}{4(d+2)}, \tag{2.47}$$

where $C_d = (d-1)S_d/(2\pi)^d$ and $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the area of unit sphere surface in d-dimensional space. Finally, using definitions (2.44) and (2.45) we arrive at the expressions for the RG-functions

$$\gamma_\nu(g) = cg + O(g^2), \quad \beta_g(g) = -2\epsilon g + 3cg^2 + O(g^3). \tag{2.48}$$

Positiveness of the coefficient c in (2.48) ensures at small $\epsilon > 0$ the existence of an infrared (IR)-stable (IR-attractive) fixed point $g_* = 2\epsilon/3c + O(g^3)$ of the RG equation (2.35) with operator (2.46) in the physical region $g > 0$. From the last of equations (2.45) the quantity γ_ν is calculated exactly, without computing diagrams: $\gamma_\nu = 2\epsilon/3$, without corrections ϵ^2, ϵ^3, etc. Explicit calculation of the constant (2.47) from diagrams is required only for checking that the coefficient c is positive, i.e. for proving the existence of IR-stable fixed point g_* in the region $g > 0$.

In the next section we give the general solution of the RG equation from which follows that if an IR-stable fixed point exists, the leading terms of IR asymptotic behavior of the connected Green functions W_n^R of any one-charge model obey the RG equation (2.35) with

the change $g \to g_*$. In the model under consideration we obtain

$$\left(\mu\partial_\mu - \gamma_v^* v\partial_v\right) W_n^R = 0\,, \tag{2.49}$$

where $\gamma_v^* = 2\epsilon$.

Canonical scale invariance is given by the equations (see general equations (2.30))

$$\left[\sum_e d_e^k e\partial_e - d_{W_n}^k\right] W_n^R = 0\,,\ \left[\sum_e d_e^\omega e\partial_e - d_{W_n}^k\right] W_n^R = 0\,, \tag{2.50}$$

where e is a set of all arguments of W_n^R, d_e and d_{W_n} are the canonical dimensions of e and W_n. Every equation of the set (2.49) (2.50) describes scaling in relation to the change of these variables, derivatives with respect to which enter into the differential operator. If we are interested in scaling when some of the variables e are fixed, we should eliminate the corresponding derivatives ∂_e by combining the available equations. Substituting required dimensions from Table 1 into (2.49), (2.50) and eliminating ∂_μ and $v\partial_v$ we arrive at the sought equation of the "critical" IR-scaling Callan-Symanczik equation

$$[-x\partial_x + \Delta_t t\partial_t + \Delta_L\partial_L - \Delta_n] W_n^R = 0 \tag{2.51}$$

with coefficients

$$\Delta_t = -\Delta_\omega = -2 + \gamma_v^*\,,\ \Delta_n = d_{W_n}^k + \Delta_\omega d_{W_n}^\omega\,,\ \Delta_L = d_L^k\,, \tag{2.52}$$

which represent the corresponding critical dimensions. Substituting the known values ($\gamma_v^* = 2\epsilon/3$, $d_{W_n}^k = -n_v + (d+1)n_{v'}$, $d_{W_n}^\omega = n_v - n_{v'}$) one obtains the following expressions for the critical dimensions (exponents)

$$\Delta_v = 1 - \frac{2\epsilon}{3},\quad \Delta_{v'} = d - \Delta_v\,,\quad \Delta_t = -\Delta_\omega = -2 + \frac{2\epsilon}{3},\quad \Delta_L = -1\,. \tag{2.53}$$

They have no corrections at order ϵ^2 and higher orders and coincide with the Kolmogorov exponents at the value $\epsilon = 2$.

2.6 SOLUTION OF RG EQUATIONS. INVARIANT VARIABLES.

In this section we demonstrate general mathematical methods how to find solutions of RG equations of type (2.35) which are typical of models under consideration. Consider

linear differential equation which is typical of RG approach

$$LF(u) = \gamma(u), \quad L = -s\partial_s + \sum_{i=2}^{n} Q_i(u)\partial_{e_i}, \tag{2.54}$$

where $s = u_1$ is the scaling parameter, $e_i = u_i, i = 2,\ldots,n$, Q_i are given functions of the parameters u and F is a sought function. The general solution of this inhomogeneous equation is the sum of its particular solution and general solution of the homogeneous equation. The latter is an arbitrary function of the full set of independent first integrals, which represent arbitrary solutions of the homogeneous equation. The number of independent first integrals is equal to the number of parameters e. It is convenient to choose first integrals $\bar{e}_i = \bar{e}_i(s,e)$, which are defined as follows

$$L\bar{e}_i(s,e) = 0, \quad \bar{e}_i(s,e)|_{s=1} = e_i\,. \tag{2.55}$$

These quantities are usually called *invariant (running) variables (charges)*.

The differential operator in the RG-equation (2.35) belongs to an important type of operators $\mathscr{D}_{\mathscr{RG}}$ defined by the equation

$$\mathscr{D}_{\mathscr{RG}}\Upsilon(s,g,a) = 0, \quad \mathscr{D}_{\mathscr{RG}} \equiv \left[-s\partial_s + \beta_g\partial_g - \sum_a \gamma_a(g)a\partial_a + \gamma(g)\right], \tag{2.56}$$

where g is the charge which defines β-function, a are other parameters ($e = \{g,a\}$) and the functions β_g, $\gamma_a(g)$ and $\gamma(g)$ are independent of s. It is possible to show, that in this special case the invariant charge $\bar{g}$ is independent of the parameters a and satisfies the differential equation known as the Gell-Mann-Low equation

$$\bar{g} = \bar{g}(s,g), \quad s\partial_s\bar{g} = \beta(\bar{g}), \quad \bar{g}|_{s=1} = g\,. \tag{2.57}$$

This equation is easily integrable

$$\log s = \int_g^{\bar{g}} \frac{dx}{\beta(x)}. \tag{2.58}$$

The last expression implicitly defines $\bar{g} = \bar{g}(s,g)$ as a function of the scale parameter s and the charge g. For models with n charges g_i ($i = 1,2,\ldots n$) one obtains a set of n equations

$$s\frac{d\bar{g}_i}{ds} = \beta_{g_i}(\bar{g}), \quad \bar{g} \equiv \bar{g}_1,\bar{g}_2,\ldots\bar{g}_n\,, \tag{2.59}$$

where $\bar{g}_i$ is a set of invariant charges with initial values equal to g_i ($\bar{g}_i|_{s=1} = g_i$). A straightforward integration (at least numerically) of these equations gives a way to find their fixed points. Instead, very often one solves the set of equations

$$\beta_{g_i^*} = 0 \tag{2.60}$$

which defines all fixed points $g_1^* \equiv g_1^*, \ldots g_n^*$. To determine the type of a fixed point one calculates the matrix $\Omega \equiv \Omega_{ik}$ at this fixed point g_*

$$\Omega_{ik} = \frac{\partial \beta_{g_i}(g)}{\partial g_k}\Big|_{g=g^*}, \quad i,k = 1,2,\ldots n\,. \tag{2.61}$$

If the matrix Ω is positive (negative) definite, then the fixed point is infrared (IR) (ultraviolet (UV)) stable. Technically one finds eigenvalues of the Ω matrix. A given fixed point is infrared (or ultraviolet) if all real parts of the eigenvalues are positive (or negative). The other invariant variables $\bar{a}$ satisfy equations

$$s\partial_s \bar{a} = -\bar{a}\gamma_a(\bar{g}), \quad \bar{a}|_{s=1} = a\,. \tag{2.62}$$

In models with one charge (coupling constant) g these equations are easily integrable and the solution has the form

$$\bar{a} = \bar{a}(s,g,a) = a\,\exp\Big(-\int_g^{\bar{g}} dx \frac{\gamma_a(x)}{\beta(x)}\Big). \tag{2.63}$$

Finally, the general solution of the equation (2.56) has the form

$$\Upsilon(s,g,a) = \Upsilon(1,\bar{g},\bar{a})\,\exp\Big(\int_g^{\bar{g}} dx \frac{\gamma(x)}{\beta(x)}\Big), \tag{2.64}$$

where $\Upsilon(1,\bar{g},\bar{a})$ is an arbitrary (scaling) function of the first integrals.

At the end of this section we will demonstrate above described procedure to solve the RG-equation for the equal-time pair correlation function of velocity field $W_2 = \langle vv\rangle$ defined by the field theoretic model with the action (2.15). W_{2R} coincides with its unrenormalized counterpart W_2. The only difference is in the choice of variables and the form of the perturbation theory (in g or g_0). In renormalized variables the correlation function W_2 depends on $\boldsymbol{k}, g, \nu, \mu$ and L. From dimensional considerations one straightforwardly obtains W_2 in the form (for simplicity we omit the transverse projection operator

$P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2$)

$$W_2 = \nu^2 k^{2-d} R(s, g, u), \quad s = k/\mu, \quad u = kL, \tag{2.65}$$

where R is arbitrary function of completely dimensionless arguments. The correlation function W_2 satisfies the general RG equation (2.35) with $\gamma_\varphi = 0$

$$D_{RG} W_2 = 0\,,\ D_{RG} = \mu\partial_\mu + \beta(g)\partial_g - \gamma_\nu(g)\nu\partial_\nu\,. \tag{2.66}$$

Using the general rule (2.64) one finds the solution of (2.66)

$$W_2 = \bar{\nu}^2 k^{2-d} R(1, \bar{g}, \bar{u})\,,\ \bar{u} = u\,, \tag{2.67}$$

where $\bar{g}$, $\bar{\nu}$ are invariant variables (first integrals of equation (2.66), see (2.55)) which depend on $s = k/\mu$ and satisfy the normalization conditions $\bar{g} = g$, $\bar{\nu} = \nu$ at $s = 1$. The invariant charge $\bar{g}$ satisfies the equation (2.58). For the invariant viscosity $\bar{\nu} = \bar{\nu}(s, \nu, g)$ using (2.63), (2.45) and (2.42) ($g\nu^3\mu^{2\epsilon} = g_0\nu_0^3 = D_0$) one arrives at the solution

$$\bar{\nu} = \nu \exp\Big(\int_{\bar{g}}^{g} dx \frac{\gamma_\nu(x)}{\beta(x)}\Big) = \left(\frac{g\nu^3}{\bar{g}s^{2\epsilon}}\right)^{1/3} = \left(\frac{D_0}{\bar{g}k^{2\epsilon}}\right)^{1/3}. \tag{2.68}$$

The invariant charge $\bar{g}$ tends to the infrared fixed point g_* as $s \to 0$ and $\bar{\nu} \to \nu_* = (D_0/g_*)^{1/3} k^{-2\epsilon/3}$, therefore at the fixed point g_* (far from the dissipation region $k \ll \mu \sim l^{-1}$, where l is Kolmogorov dissipative length [15]) the equal-time pair correlation function of the velocity field acquires the scaling form

$$W_2 = \left(\frac{D_0}{g_*}\right)^{2/3} k^{2-d-4\epsilon/3} R(1, g_*, kL). \tag{2.69}$$

For the value $\epsilon = 2$ we again obtain the kinetic energy spectrum $E(k) = W_2 k^{d-1}$ in a power form with the Kolmogorov exponent −5/3. The scaling function R remains unknown and, in general, can be calculated only perturbatively as an infinite series in powers of the parameter ϵ. It is quite a general situation which is common for the theory of critical phenomena, theory of developed turbulence and nonlinear stochastic dynamics as whole. But in the most interesting asymptotic region $kL \gg 1$ (inertial range) this trivial expansion fails and it is necessary to elaborate an algorithm to find the asymptotic behavior of the scaling function R.

3 FIELD-THEORETIC REPRESENTATION OF THE MASTER EQUATION

3.1 INTRODUCTION

The irreversible annihilation reaction $A + A \to \varnothing$, also known as the mutually annihilating random walk, is a fundamental model of non-equilibrium physics. The particles A perform chaotic motion due to diffusion and after the mutual collision they may react with constant microscopic probability K_0 per unit time. Usually it is assumed that resulting molecule $\varnothing$ is inert, i.e. chemically inactive and without any backward influence on the motion of reacting A particles. Many reactions of this type are observed in diverse chemical, biological or physical systems. For instance various models such as formation of domain in some magnetic materials [54], annihilation of excitons in crystals [55] or model for spreading of opinion of voters in one dimension [11] can be described in terms of annihilation process.

The usual approach to the problems dealing with chemical reactions is based on the use of the kinetic rate equation [13] for concentration $n(t, \mathbf{x})$. It leads to a self-consistent description analogous to the mean-field approximation in the theory of critical phenomena in the sense that fluctuations in the concentration are neglected. In other formulation one can assume that the particle concentration is spatially homogeneous $n = n(t)$. This homogeneity can be thought as a consequence of either very high mobility of the reactants or of a very small probability that a chemical reaction actually occurs when reacting particles collide. For the annihilation process $A + A \to \varnothing$ kinetic rate equation can be formulated as

$$\partial_t n(t) = -K_0 n^2(t), \tag{3.1}$$

which can be easily integrated and the obtained solution is

$$n(t) = \frac{n_0}{1 + n_0 K_0 t}, \tag{3.2}$$

where $n_0 \equiv n(0)$ is initial number of particles. This equation predicts a long-time ($t \to \infty$) asymptotic decay behaving as $n(t) \sim t^{-1}$ without any dependence on the value of space dimension. This is a common situation observed in the mean field-like theories. In what follows we will refer to the value of time exponent α in power law dependence for concentration $n(t) \propto t^{-\alpha}$ as *decaying exponent*. Note also that long time behavior doesn't dependent on the initial number n_0 of reacting particles. On the other hand, if the particle mobility becomes sufficiently small, or equivalently, if the microscopic reaction probabil-

ity K_0 becomes large enough (so particles react immediately after mutual collision) there is a possible transition to a new regime. In it is more probable that the given particle reacts with particles in its neighbourhood than with distant ones. This behavior is known as the diffusion-controlled regime [13, 23]. To gain a physical insight let's consider diffusion process (also known as continuos random walk) to be responsible for the motion of particles. The well known property of diffusion [25] is re-entrancy of the visited sites in low space dimensions. In the concrete for $d = 1$ (line) or $d = 2$ (plane) the probability that diffusing particle will ever return ($t \to \infty$) to the starting point is equal 1. Physically it means the diffusion particle sweeps very well its local neighbourhood and thus it is very probable that it will react with other particle in its vicinity. Hence it is reasonable to expect that after short amount of time the system would be in a state where there is a lot of isolated particles, that need effectively longer time to traverse to each other and hence to annihilate. This mechanism can effectively slow down time evolution of the process and thus lower the decaying exponent to other value than 1 predicted by equation (3.2). Its approximate value can be guessed according to following scaling argument. The re-entrancy property leads to the scaling relation

$$V(t) \sim r^d(t)\,, \tag{3.3}$$

where $\sim$ denotes correspondig scaling relation between physical quantities. The mean root square for diffusing particle scales as $r(t) \sim (Dt)^{1/2}$ and therefore the mean particle number should behave as

$$n(t) \propto \frac{1}{V(t)} \sim \frac{1}{t^{d/2}} = \frac{1}{t^{1+\Delta}}\,, \tag{3.4}$$

where exponent Δ denotes deviation from the space dimension 2 via relation

$$d = 2 + 2\Delta\,. \tag{3.5}$$

For space dimension $d = 3$ we have $V(t) \sim t$, because now the diffusing particle explore always new volume and reentrance can be effectively neglected. Therefore the same behavior as the one described by (3.1) would be observed. From these simple analysis it could be estimated that space dimension $d_c = 2$ is upper critical dimension for annihilation process, above which mean field approximation is valid. The more rigorous treatment [24] proves this conclusion and also shows logarithmic correction for $n(t)$ at critical dimension, which couldn't be determined by simple scaling analysis.

3.2 MASTER EQUATION IN FOCK SPACE

In our work we are mainly interested in the annihilation process $A + A \xrightarrow{K_0} \varnothing$, therefore in this section we describe possible derivation of the field-theoretic model for such process, which allows to take into account spatial inhomogenities and randomness in individual reaction events. Probably the most fundamental description of reaction processes is based on the use of master equation [13]. We summarize general principles of how such master equation can be mapped onto suitable Fock space, which makes it possible to use very powerful methods of quantum field theory. The resulting action functional can be treated systematically by methods as Feynman diagrammatic technique, renormalization group and operator product expansion [2]. Here we focus mainly on the field-theoretic description of diffusion and reaction process, but it is possible [56] to generalize this approach to include effects as multiple reaction schemes, disorder effects, influence of spatial boundaries etc.

Let's start with the particles on a regular, infinite, hypercubic lattice with lattice spacing a in d-dimensional space. The sites of lattice can be labeled by natural numbers ($i = 1, 2, \ldots$). The A particles are performing continuous random walk on this lattice (random hopping betweeen adjoint sites) with diffusion constant D/a^2 (the factor a^2 will be eliminated in the continuum limit). It is assumed that reaction process can happen only for particles that occupy the same lattice site with the probability rate K_0. Complete (microscopic) description of such stochastic problem can be given in terms of evolution equations for probabilities $P(t;\{n\})$, where $\{n\}$ is given microstate $\{n_1, n_2, \ldots\}$ characterised by n_1 particles at site 1 , n_2 at site 2 and so on. These equations (known as master equation) express the balance between incoming and outcoming probabilities [13] for given state $\{n\}$ and in the compact notation they can be written as

$$\frac{dP(t;\{n\})}{dt} = \sum_{\{m\}} R_{m\to n} P(t;\{n\}) - \sum_{\{m\}} R_{n\to m} P(t;\{n\}), \tag{3.6}$$

where $R_{m\to n}$ is the transition probability rate from state m to the state n. According to the work of Doi [26] (see also [57]) such system of coupled differential equations (3.6) can be rewritten in terms of creation and annihilation operators well known from quantum mechanics. If there is no site occupation restriction, bosonic operators for each lattice site i can be introduced with the following commutation relations

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \quad [\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0. \tag{3.7}$$

The ground state $|0\rangle$ is defined as

$$\hat{a}_i|0\rangle = 0 \quad \text{for all sites } i, \tag{3.8}$$

which corresponds to the empty lattice (without any A particle). From bosonic commutation relations (3.7) also follows important relations

$$\hat{a}_i^n \hat{a}_i^\dagger = n\hat{a}_i^{n-1} + \hat{a}_i^\dagger \hat{a}^n, \quad \hat{a}_i \hat{a}_i^{n\dagger} = n\hat{a}_i^{(n-1)\dagger} + \hat{a}^{n\dagger}\hat{a}_i \,. \tag{3.9}$$

The state $|\{n\}\rangle$ with given lattice configuration $\{n\} = \{n_1, n_2, \ldots\}$ is introduced with a different normalization in comparison with one used for second quantization method

$$|\{n\}\rangle = \hat{a}_1^{\dagger n_1} \hat{a}_2^{\dagger n_2} \ldots |0\rangle. \tag{3.10}$$

Using relations (3.9) it could be directly shown that

$$\begin{aligned} \hat{a}_i|\{n\}\rangle &= n_i|\{n_1, n_2, \ldots, n_i - 1, \ldots\}\rangle, \\ \hat{a}_i^\dagger|\{n\}\rangle &= |\{n_1, n_2, \ldots, n_i + 1, \ldots\}\rangle, \\ \hat{a}_i^\dagger \hat{a}_i|\{n\}\rangle &= n_i|\{n\}\rangle, \end{aligned} \tag{3.11}$$

where the last relation legitimizes identification

$$\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i \tag{3.12}$$

for the number operator at site i. The scalar product between two states $|\{n\}\rangle$ and $|\{m\}\rangle$ can be obtained

$$\langle\{n\}|\{m\}\rangle = \prod_{i=1} \delta_{n_i,m_i} n_i!, \tag{3.13}$$

where $\delta_{i,j}$ stands for Kronecker symbol and $n! = 1 \times 2 \times \ldots n$ is factorial function. The complete information about stochastic system is embodied in the probability $P(t;\{n\})$ and in the Doi formalism it is incorporated into the state vector $|\Phi\rangle$, that is defined as follows

$$|\Phi(t)\rangle \equiv \sum_{\{n\}} P(t;\{n\})|\{n\}\rangle = \sum_{\{n\}} P(t;\{n\})\, \hat{a}_1^{\dagger n_1} \hat{a}_2^{\dagger n_2} \ldots |0\rangle, \tag{3.14}$$

where the sum runs over all possible lattice occupations. Now the task is to rewrite mas-

ter equation (3.6) into the Schrödinger-like form for the state vector $|\Phi\rangle$

$$\frac{d}{dt}|\Phi(t)\rangle = -\hat{H}|\Phi(t)\rangle, \tag{3.15}$$

with some "Hamiltonian" $\hat{H}$, whose exact form depends on the system under consideration. Then the equation (3.15) can be formally integrated to obtain $|\Phi(t)\rangle = \mathrm{e}^{-\hat{H}t}|\Phi(0)\rangle$. The initial state $|\Phi(0)\rangle$ has to be specified for full description. In the case of chemical reactions initial distribution of particles $P(0;\{n\})$ is usually prescribed and initial state follows from the definition (3.14). From the technical point of view the most convenient choice for the singe-species annihilation reaction is Poisson distribution. Unlike for bimolecular reaction [58] the long-time behavior is independent from the concrete form of initial conditions.

Let's derive diffusion part $\hat{H}_D$ of Hamiltonian H, which corresponds to the diffusive (random walk) movement of A particles. First consider two-site system with n_1 particles at site 1 and n_2 particles at site 2 with one-directional hopping process $1 \to 2$ at the rate D_0/a^2. For such process master equation (3.6) should be given as

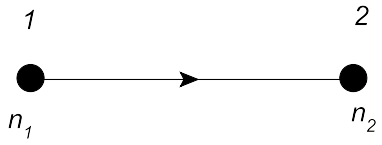


Figure 4: Hopping process

$$\frac{dP(n_1,n_2)}{dt} = \frac{D_0}{a^2}(n_1+1)P(n_1+1,n_2-1) - \frac{D_0}{a^2}n_1P(n_1,n_2), \tag{3.16}$$

where factors n_1+1 and n_1 are combinatorial factors resulting from the fact, that A particles jump independently from each other. Multiplying both sides of the equation (3.16) by the term $a_1^{\dagger n_1}a_2^{\dagger n_2}$ and performing sum $\sum_{\{n_1,n_2\}}$ over all possible sites' occupations we arrive at

$$\frac{d|\Phi\rangle}{dt} = \frac{D_0}{\hat{a}^2}(\hat{a}_2^\dagger\hat{a}_1 - \hat{a}_1^\dagger\hat{a}_1)|\Phi\rangle. \tag{3.17}$$

This result is easily generalized to the both-directional case $1 \leftrightarrow 2$ described by the following master equation

$$\frac{d|\Phi\rangle}{dt} = -\frac{D_0}{a^2}(\hat{a}_2^\dagger - \hat{a}_1^\dagger)(\hat{a}_2 - \hat{a}_1)|\Phi\rangle \tag{3.18}$$

and hence the diffusion part H_D between two given sites can be written as

$$\hat{H}_D = \frac{D_0}{a^2}(\hat{a}_2^\dagger - \hat{a}_1^\dagger)(\hat{a}_2 - \hat{a}_1). \tag{3.19}$$

Now consider annihilation process at given site 1 and derive corresponding part H_R of Hamiltonian. Because any two particles can react together we can write

$$\frac{dP(n)}{dt} = K_0(n+2)(n+1)P(n+2) - K_0 n(n-1)P(n), \tag{3.20}$$

where again combinatorial factors are taken into account. After short algebraic manipulations and use of the relations (3.7) and (3.11) it can be rewritten into the Doi formalism

$$\frac{d|\phi\rangle}{dt} = K_0(\hat{a}^2 - \hat{a}^{\dagger 2}\hat{a}^2)|\phi\rangle\,, \tag{3.21}$$

from which we deduce that reaction part of Hamiltonian equals

$$\hat{H}_R = -K_0(\hat{a}^2 - \hat{a}^{\dagger 2}\hat{a}^2)\,. \tag{3.22}$$

Results (3.19) and (3.22) are easily generalized to include all the lattice sites. Consequently the total Hamiltonian that accounts for diffusion and annihilation process on the hypercubic lattice has the following form

$$\hat{H}_D + \hat{H}_R = \frac{D_0}{a^2}\sum_{<ij>}(\hat{a}_i^\dagger - \hat{a}_j^\dagger)(\hat{a}_i - \hat{a}_j) - K_0\sum_i(\hat{a}_i^2 - \hat{a}_i^{\dagger 2}\hat{a}_i^2)\,, \tag{3.23}$$

where the first sum runs over the neighbouring sites i and j. Non-Hermitian Hamiltonians such as one given in (3.23) are often observed in the case of out-of-equilibrium systems [11], that couldn't be obtained as dynamical counterpart of some static model. Non-hermiticity also means that reactions rates in (3.6) doesn't satisfy the detailed balance condition and thus equilibrium couldn't be characterised by Gibbs distribution. Doi formalism also exhibits other differences with usual quantum mechanics. They are caused by the fact that physical observables cannot be given as bilinear products $\langle\Phi|A|\Phi\rangle$, since according to (3.14) this would imply expressions bilinear in probability $P(t;\{n\})$.

Now let's take a closer look at possible derivation of the ensemble average value for quantity A within Doi approach. It is physically appropriate to assume that A can be expressed as function of the occupation numbers $A = A(\{n\})$. Examples of such quantities that interesting for the case of chemical reactions are

(a) mean particle number (concentration)

$$n \longleftrightarrow \sum_i \hat{a}_i^\dagger \hat{a}_i \tag{3.24}$$

(b) two-point correlation function (between sites i and j)

$$C(i,j) \longleftrightarrow \hat{a}_i^\dagger \hat{a}_i \hat{a}_j^\dagger \hat{a}_j \tag{3.25}$$

The ensemble average of A is then clearly given by the expression

$$\langle A(t)\rangle = \sum_{\{n\}} P(t;\{n\}) A(\{n\}) \tag{3.26}$$

and from the technical point of view it would be very convenient to have projection state $\langle \mathscr{P}|$ such that following identity is valid

$$\langle A(t)\rangle = \sum_{\{n\}} P(t;\{n\}) \langle \mathscr{P}|\hat{A}(\{\hat{a}^\dagger,\hat{a}\})\hat{a}_1^{\dagger n_1}\hat{a}_2^{\dagger n_2}\ldots|0\rangle = \langle \mathscr{P}|\hat{A}|\Phi(t)\rangle. \tag{3.27}$$

Substituting the formal solution of "Schrödinger" equation (3.15) leads to yet another form of the last expression

$$\langle A(t)\rangle = \langle \mathscr{P}|\hat{A}\exp(-\hat{H}t)|\Phi(0)\rangle. \tag{3.28}$$

Here the operator $\hat{A}(\{\hat{a}^\dagger,\hat{a}\})$ is obtained from the classical function $A(\{n\})$ with the substitution $n_i \to \hat{a}_i^\dagger \hat{a}_i$ at every site i, what is justified by (3.12). Hence by comparing (3.26) and (3.27) it is easy to guess that the projection state $\langle \mathscr{P}|$ should satisfy relation

$$\langle \mathscr{P}|\hat{A}(\{a^\dagger,a\})a_1^{\dagger n_1}a_2^{\dagger n_2}\ldots|0\rangle = A(\{n\})\langle \mathscr{P}|a_1^{\dagger n_1}a_2^{\dagger n_2}\ldots|0\rangle = A(\{n\})\,. \tag{3.29}$$

It implies that the following two conditions

$$\langle \mathscr{P}|\hat{a}_i^\dagger = \langle \mathscr{P}| \quad \text{for every site } i, \quad \langle \mathscr{P}|0\rangle = 1 \tag{3.30}$$

has to be valid for operator $\mathscr{P}$. By direct use of relations (3.7) we can conclude that the following choice of projection state

$$\langle \mathscr{P}| = \langle 0|\exp\Big(\sum_i \hat{a}_i\Big) \tag{3.31}$$

serves our purpose. Note that if operator $\hat{A}$ is written in normal order (all creation operators are commuted to the left), it can then be written in terms of annihilation operators a_i only using properties (3.30) of projection state $\langle\mathscr{P}|$. The operator corresponding to (3.24) is then simply $\hat{a}_i$, while the correlation function (3.25) corresponds to operator $\hat{a}_i\hat{a}_j+\delta_{ij}\hat{a}_j$. For the technical reasons it is reasonable to commute the factor $\mathrm{e}^{\sum_i \hat{a}_i}$ to the right in equation (3.27). In order to do this we employ the following formula

$$\mathrm{e}^{\hat{a}_i}\hat{a}_i^\dagger = (\hat{a}_i^\dagger+1)\mathrm{e}^{\hat{a}_i}, \tag{3.32}$$

which is derived from relations (3.9) and thus equation (3.28) can be rewritten into the form

$$\langle A(t)\rangle = \left\langle 0\left|\hat{A}(\{\hat{a}^\dagger+1,\hat{a}\})\exp(-\hat{H}(\{\hat{a}^\dagger+1,\hat{a}\})t)\right|\mathrm{e}^{\sum_i \hat{a}_i}\Phi(0)\right\rangle, \tag{3.33}$$

where the substitution $\hat{a}_i^\dagger \to \hat{a}_i^\dagger+1$ was performed both in the expression for operator $\hat{A}$ and also in the original Hamiltonian $\hat{H}(\{a^\dagger,a\})$. For further use let us write expression for the mean particle number (3.24)

$$n(t) = \left\langle 0\left|\sum_i a_i \exp(-\hat{H}(\{a^\dagger+1,a\})t)\right|\mathrm{e}^{\sum_i a_i}\Phi(0)\right\rangle. \tag{3.34}$$

In the case of annihilation process described by the equation (3.23) thus we obtain

$$\hat{H}_D+\hat{H}_R = \frac{D_0}{a^2}\sum_{<ij>}(a_i^\dagger-a_j^\dagger)(a_i-a_j)+K_0\sum_i(2a_i^\dagger a_i^2+a_i^{\dagger 2}a_i^2). \tag{3.35}$$

As was mentioned above suitable choice for the initial condition is the Poisson distribution, that for a given site i corresponds to

$$p(n_i) = \mathrm{e}^{-n_0}\frac{n_0^{n_i}}{n_i!}, \tag{3.36}$$

where n_0 stands for mean particle number. Using definition (3.14) it leads to the initial state vector in the form

$$|\Phi(0)\rangle \quad = \quad \sum_{\{n_1\}}\mathrm{e}^{-n_0}\frac{n_0^{n_1}}{n_1!}\hat{a}_1^{\dagger n_1}\sum_{\{n_2\}}\mathrm{e}^{-n_0}\frac{n_0^{n_2}}{n_2!}\hat{a}_2^{\dagger n_2}\ldots|0\rangle = \prod_i \mathrm{e}^{-n_0}\mathrm{e}^{n_0\sum_i \hat{a}_i^\dagger}|0\rangle. \tag{3.37}$$

We see that after substitution $\hat{a}^\dagger \to \hat{a}^\dagger+1$ term e^{-n_0} drops out.

3.3 CONTINUUM LIMIT

In the field of critical phenomena the emphasis often lies in analysis and determination of possible behavior of studied system. It turns out that near its second-order phase transition [10] a whole set of different models behave in the same way. Despite the fact that they describe different physical systems, they can exhibit the same behavior concerning so called universal quantities. They are model-independent, but could depend on universal parameters such as space dimension, number of components of order parameter or symmetries of the system. Examples of such universal quantities are critical exponents [59], describing singular behavior of various functions.

Now we summarize main points for the derivation of continuum limit for the Hamiltonian (3.23), which allows us to study universal properties of the annihilation process $A + A \to \varnothing$ around its critical dimension. Here we just briefly note few important steps following [56]. The main task consists in evaluation of matrix elements for the evolution operator $\exp(-\hat{H}(\{a^\dagger, a\})t)$. In order to do this we apply Trotter formula [60] according to which exponential $\mathrm{e}^{-\hat{H}t}$ can be written as infinite product

$$\exp(-\hat{H}t) = \lim_{\Delta t \to 0} (1 - \hat{H}\Delta t)^{t/\Delta t} = (1 - \hat{H}\Delta t)(1 - \hat{H}\Delta t)\dots \tag{3.38}$$

Here we assume that $N\Delta t = t$ and at the end of our derivation we let the number of time slices $N \to \infty$ (or equivalently $\Delta t \to 0$). Now into each time slice complete set of coherent states is inserted, which results into mapping of operators $\hat{a}_i^\dagger, \hat{a}_i$ onto complex numbers. Coherent states are explicitly defined as [60]

$$|\psi\rangle = \exp\Big(-\frac{1}{2}|\psi|^2 + \psi\hat{a}^\dagger\Big)|0\rangle \tag{3.39}$$

and they form the eigenstate basis of the annihilation operator

$$\hat{a}|\psi\rangle = \psi|\psi\rangle, \quad \langle\psi|\hat{a}^\dagger = \psi^*\langle\psi|, \tag{3.40}$$

where star stands for complex conjugation. The important property of coherent state is their overlap function between different eigenstates

$$\langle\psi_1|\psi_2\rangle = \exp\Big(-\frac{1}{2}|\psi_1|^2 - \frac{1}{2}|\psi_2|^2 + \psi_1^*\psi_2\Big). \tag{3.41}$$

For a single site we can write identity in the form

$$1 = \sum_n \frac{1}{n!}|n\rangle\langle n| = \sum_{m,n} \frac{1}{n!}|n\rangle\langle m|\delta_{mn} = \int \frac{d\psi^* d\psi}{\pi}|\psi\rangle\langle\psi|, \tag{3.42}$$

where the orthogonal relation

$$\delta_{mn} = \frac{1}{\pi m!}\int d\psi^* d\psi \ \exp(-|\psi|^2)\psi^{*m}\psi^n \tag{3.43}$$

has been used (note the apperance of the weight function $\mathrm{e}^{-|\psi|^2}$). Generalization of (3.42) to the whole lattice is straightforward

$$1 = \int \prod_i \frac{d\psi^*_{(i,j)} d\psi_{(i,j)}}{\pi}\Big|\{\psi\}_j\Big\rangle\Big\langle\{\psi\}_j\Big|, \tag{3.44}$$

where now $\{\psi\}_j = (\psi_{(1,j)}, \psi_{(2,j)}, \ldots)$ denotes set of all eigenvalues corresponding to the annihilation operators $\hat{a}_i$ at each lattice site at time $j\Delta t$ ($j = 0, \ldots, N$). Inserting (3.44) into each time slice in (3.38) we get

$$\exp(-\hat{H}t) = \frac{1}{\mathcal{N}} \lim_{\Delta t\to 0}\int [d\psi^*][d\psi]\Big|\{\psi\}_N\Big\rangle\Big(\prod_{j=1}^{N}\Big\langle\{\psi\}_j\Big|\exp(-\hat{H}(\{\hat{a}^\dagger, \hat{a}\})\Delta t)\Big|\{\psi\}_{j-1}\Big\rangle\Big)\Big\langle\{\psi\}_0\Big|, \tag{3.45}$$

where $\mathcal{N}$ is the normalization constant and we have introduced notation

$$[d\psi^*][d\psi] \equiv \prod_{i,j} d\psi^*_{(i,j)} d\psi_{(i,j)} \tag{3.46}$$

for functional measure. Note that if we deal with normal ordered Hamiltonian (as is the case,e.g. for (3.35)) using relations (3.40) we can immediately write down

$$\Big\langle\{\psi\}_j\Big|\exp(-H(\{a^\dagger, a\})\Delta t)\Big|\{\psi\}_{j-1}\Big\rangle = \Big\langle\{\psi\}_j\Big|\{\psi\}_{j-1}\Big\rangle\exp(-H(\{\psi^*\}_j, \{\psi\}_{j-1})\Delta t), \tag{3.47}$$

where $H(\{\psi^*\}_j, \{\psi\}_{j-1})$ is easily obtained by replacement $\hat{a}_i \to \psi_i, \hat{a}_i^\dagger \to \psi_i^*$. The remaining term in (3.47) more precisely stands for the expression

$$\Big\langle\{\psi\}_j\Big|\{\psi\}_{j-1}\Big\rangle = \prod_i\Big\langle\psi_{(i,j)}\Big|\psi_{(i,j-1)}\Big\rangle, \tag{3.48}$$

that using overlap relation (3.41) can be rewritten as

$$\left\langle \psi_{(i,j)} \middle| \psi_{(i,j-1)} \right\rangle = \exp\left(-\psi^*_{(i,j)}[\psi_{(i,j)} - \psi_{(i,j-1)}]\right)\exp\left(\frac{1}{2}|\psi_{(i,j)}|^2 - \frac{1}{2}|\psi_{(i,j-1)}|^2\right). \tag{3.49}$$

The whole scalar product from (3.45) can be expressed to the first order in time as

$$\prod_j \left\langle \psi_{(i,j)} \middle| \psi_{(i,j-1)} \right\rangle = \exp\left(-\sum_j \psi^*_{(i,j)} \frac{d\psi_{(i,j)}}{dt}\Delta t + O(\Delta t)\right)\exp\left(\frac{1}{2}|\psi_{(i,N)}|^2 - \frac{1}{2}|\psi_{(i,0)}|^2\right), \tag{3.50}$$

so in the continuum time limit $\Delta t \to 0$ we obtain functional integral representation for the evolution operator in the form

$$\exp(-\hat{H}t) = \int \frac{[d\psi^*][d\psi]}{\mathcal{N}} \left|\{\psi\}_N\right\rangle \exp\left(\frac{1}{2}|\psi_i(t)|^2 - \frac{1}{2}|\psi_i(0)|^2 - \int_0^t dt[\psi_i^* \partial_t \psi_i + H(\{\psi^*, \psi\})]\right) \left\langle \{\psi\}_0 \right|. \tag{3.51}$$

In the calculation of mean values such as (3.24) and (3.25) we have to act on the expression (3.51) from the left by projection state $\langle \mathscr{P}|$ and from right by initial state $|\Phi(0)\rangle$. It could be done in the following manner: first we note that the following relation holds

$$\left\langle \{\psi\}_0 \middle| \Phi(0) \right\rangle \prod_i \exp\left(-\frac{1}{2}|\psi_i(0)|^2\right) = \exp\left(\sum_i [n_0 \psi_i^*(0) - |\psi_i(0)^2|]\right), \tag{3.52}$$

where we have used initial state in the form (3.37) and relations (3.39,3.40). Using equation (3.32) one can proceed as follows

$$\begin{aligned}
\left\langle \mathscr{P} \middle| \{\psi\}_N \right\rangle \prod_i \exp\left(\frac{1}{2}|\psi_i(t)|^2\right) &= \left\langle 0 e^{\sum_i \hat{a}_i} \exp \sum_i (-\frac{1}{2}|\psi_i(t)|^2 + \psi_i(t)\hat{a}_i^\dagger) \middle| 0 \right\rangle \prod_i \exp\left(\frac{1}{2}|\psi_i(t)|^2\right) \\
&= \left\langle 0 \middle| \prod_i \exp(\hat{a}_i) \exp(\psi_i(t)\hat{a}_i^\dagger) \middle| 0 \right\rangle \\
&= \left\langle 0 \middle| \prod_i \sum_{k=0} \frac{\psi_i^k(t)}{k!} \exp(\hat{a}_i) \hat{a}_i^{\dagger k} \middle| 0 \right\rangle \\
&= \left\langle 0 \middle| \prod_i \exp\left(\psi_i(t)(\hat{a}_i^\dagger + 1)\right) \exp(\hat{a}_i) \middle| 0 \right\rangle \\
&= \exp\left(\sum_i \psi_i(t)\right).
\end{aligned} \tag{3.53}$$

Putting together terms from (3.51-3.53) and inserting them into expression (3.28) for the mean value of quantity A we arrive at important expression

$$\langle A(t) \rangle = \mathcal{N}^{-1} \int [d\psi^*][d\psi] A(\{\psi\}) \exp[S(\{\psi^*, \psi\})] \tag{3.54}$$

with action functional S given as

$$S(\{\psi^*,\psi\}) = \sum_i \Big(\psi_i(t) + n_0\psi_i^*(0) - |\psi_i(0)|^2 - \int_0^t dt[\psi_i^*\partial_t\psi_i + H(\{\psi^*\},\{\psi\})].\Big) \tag{3.55}$$

The normalization constant $\mathcal{N}$ is now fixed by the condition

$$\mathcal{N} = \int [d\psi^*][d\psi]\exp[S(\{\psi^*,\psi\})] \tag{3.56}$$

Before taking the continuum limit in space let us note that the initial term $-|\psi_i(0)|^2$ in (3.55) can actually be dropped (see section 3.3 in [56]). Continuum limit then can be performed in traditional manner according to the substitution

$$\sum_i \to \frac{\int d\mathbf{x}}{a^d}, \quad \psi_i \to \psi(t,\mathbf{x})a^d, \quad \psi_i^*(t) \to \psi^\dagger(t,\mathbf{x}), \quad n_0 \to n_0 a^d, \tag{3.57}$$

that leads to the field-theoretic action for the scalar fields $\psi^\dagger(\mathbf{x},t)$ and $\psi(\mathbf{x},t)$

$$S = -\int_0^\infty dt\int d\mathbf{x}\Big\{\psi^\dagger\partial_t\psi - D_0\psi^\dagger\nabla^2\psi - \lambda_0[1-\psi^{\dagger 2}]\psi^2\Big\} + \int d\mathbf{x}\Big[\psi(t,\mathbf{x}) + n_0\psi^\dagger(0,\mathbf{x})\Big]. \tag{3.58}$$

It corresponds to the lattice Hamiltonian (3.23). After the shift $\psi^\dagger \to \psi^\dagger + 1$ we arrive at the desiring result, which is the the field theoretic action S for the annihilation reaction $A + A \to \varnothing$

$$S = -\int_0^\infty dt\int d\mathbf{x}\Big\{\psi^\dagger\partial_t\psi - D_0\psi^\dagger\nabla^2\psi + \lambda_0 D_0[2\psi^\dagger + (\psi^\dagger)^2]\psi^2\Big\} + n_0\int d\mathbf{x}\ \psi^\dagger(\mathbf{x},0). \tag{3.59}$$

which corresponds to the continuum limit of the action used in (3.35) and is needed for calculation physical quantities as the mean value (3.33), that can be calculated as functional integral

$$\langle A(t)\rangle = \int \mathscr{D}\psi^\dagger\mathscr{D}\psi A\Big\{[\psi^\dagger(t)+1]\psi(t)\Big\}\mathrm{e}^{\mathrm{S}}. \tag{3.60}$$

Here $\mathscr{D}\psi^\dagger\mathscr{D}\psi$ is the continuum functional measure. The continuum limit of the expression (3.34) for mean particle number becomes

$$n(t,\mathbf{x}) = \Big\langle 0\Big|\psi(\mathbf{x})\exp\Big(-H\{[\psi^\dagger(t)+1]\psi(t)\}t\Big)\exp\Big(n_0\int d\mathbf{x}\ \psi^\dagger\Big)\Big|0\Big\rangle. \tag{3.61}$$

4 FIELD-THEORETIC STUDY OF REACTION PROCESS $A + A \to \varnothing$

4.1 INTRODUCTION

A typical reaction usually occurs in liquid or gaseous environment. Thermal fluctuations of this underlying environment or some external advection field such as atmospheric eddies could have additional influence on motion of the reacting particles. Therefore, it is interesting to study what effect external velocity field can have on the annihilation process. Theoretical analysis has been already performed in the case of time-independent random drift in Refs. [61, 62, 63, 64]. In Ref. [51] annihilation process $A + A \to \varnothing$ was investigated near its critical dimension $d_c = 2$ with the time-dependent random drift, based on the use of the stochastic Navier-Stokes equation. From the point of the Navier-Stokes equation the situation in low space dimension is even more intriguing. It is well-known fact [15] that in the case of space dimension $d = 2$, there is inviscid conservation law of enstrophy, which is absent in the three dimensional case. And any such law could have a profound influence on the large scale behavior. Calculations in Ref. [51] were performed in the one-loop approximation. As may be readily seen from examination of the Feynman graphs, in the one-loop approximation there is no influence of the velocity fluctuations on the renormalization of the interaction vertices. Therefore the influence of the velocity could be only caused only through the renormalization of the diffusion term. However, higher order terms of the perturbation series can have significant effect on the critical properties and thus could cause a profound change of them. In this section we describe a model for reactive scalar field advected by random velocity field. Velocity field will be generated by the stochastic Navier-Stokes equation, which could be used for production of velocity field corresponding to thermal fluctuations [45, 65] and also a turbulent velocity field with the Kolmogorov scaling behavior [31]. A powerful tool for analyzing asymptotic behavior of stochastic systems is provided by the renormalization-group (RG) method. It allows to determine long-time – or infra-red (IR) – asymptotic regimes of the system and also it is very efficient tool for calculation of various universal physical quantities, e.g. critical exponents. The aim of this study is to examine the IR behavior of the annihilation process under the influence of advecting velocity fluctuations and to determine its stability. Mapping procedure based on the Doi formalism [26] an effective field-theoretic model for the annihilation process was described in detail in the previous section. The RG method is applied on this model and within the two-parameter expansion the renormalization constants and fixed points of the renormalization group are determined in

the two-loop approximation. The non-linear integro-differential equation, which includes first non-trivial corrections to the (3.1), is obtained for the mean particle number and it is shown how the information about IR asymptotics can be extracted from it.
This chapter is organized as follows. In Sec. 4.2 the main ingredients of the field-theoretic model for the annihilation process are summarized together. It is also shown how both the Kolmogorov scaling and thermal fluctuations can be included into the model. The ultraviolet (UV) renormalization of the model and elaborated algorithm for the calculation of the renormalization constants is described in Sec. 4.3. Fixed points of the RG are classified together with their stability regions and possible scaling regimes are presented in Sec 4.4. In Sec. 4.5 the integro-differential equation for the mean particle number is derived and analysis of its solution is given. Conclusions are presented in Sec. 4.6.

4.2 FIELD-THEORETIC MODEL OF ANNIHILATION PROCESS

Let us study anomalous kinetics of the general type of the irreversible single-species annihilation reaction

$$A + A \xrightarrow{K_0} \varnothing, \tag{4.1}$$

with the unrenormalized (mean field) rate constant K_0. The action for this type of reaction is given by the (3.59). Influence of advection can be included [51, 21] through the following replacement of time derivative term

$$\partial_t \to \nabla_t \equiv \partial_t + (\mathbf{v}.\boldsymbol{\nabla}). \tag{4.2}$$

The total action for the annihilation process in the presence of advection can therefore be written as

$$\begin{aligned} S_1 &= -\int_0^\infty dt \int d\mathbf{x}\{\psi^\dagger \partial_t \psi + \psi^\dagger \boldsymbol{\nabla}(\mathbf{v}\psi) - D_0 \psi^\dagger \nabla^2 \psi + \lambda_0 D_0 [2\psi^\dagger + (\psi^\dagger)^2]\psi^2 + \\ &\quad n_0 \int d\mathbf{x} \psi^\dagger(\mathbf{x}, 0)\}. \end{aligned} \tag{4.3}$$

Because of dimensional reasons we have introduced dimensionless parameter λ_0

$$K_0 = \lambda_0 D_0 \tag{4.4}$$

and λ_0 will be another charge of theory. In order to analyze the effect of velocity fluctuations on the reaction process it is necessary to average the expectation value (3.60)

now with action (4.3) over all possible realizations of velocity fluctuations **v**. The most realistic description of the velocity field $\mathbf{v}(x)$ is based on the use of the stochastic Navier-Stokes equation (2.1). In contrast to the case of three dimensional turbulence (2.4) now the injection function has to be chosen in the form

$$d_f(k) = g_{10}\nu_0^3 k^{4-d-2\epsilon} + g_{20}\nu_0^3 k^2. \tag{4.5}$$

The nonlocal term is often used to generate the turbulent velocity field with Kolmogorov's scaling [30, 31, 1] and it corresponds to the value $\epsilon = 2$. It could be shown [36] that in two dimensional turbulence the UV divergences doesn't appear only for the Green function $\langle v\tilde{v}\rangle$, but also for the function $\langle \tilde{v}\tilde{v}\rangle$. Together with the correct use of UV renormalization this fact leads to the generalization of the injection function (2.4) [49] to make the model multiplicatively renormalizable. The local term $g_{20}\nu_0^3 k^2$ has been added not only because of renormalization reasons but has also an important physical meaning. Such a term in the force correlation function describes generation of thermal fluctuations of the velocity field near equilibrium [45, 65] and thus can mimic the usual environment in which chemical reactions take place.

Averaging (3.60) over the random velocity field **v** is done with the "weight" functional $\mathcal{W}_2 = \mathrm{e}^{S_2}$, where S_2 is the effective action for the advecting velocity field

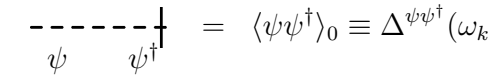
$$S_2 = \frac{1}{2}\int dtd\mathbf{x}d\mathbf{x}'\ \tilde{\mathbf{v}}(\mathbf{x},t).\tilde{\mathbf{v}}(\mathbf{x}',t)d_f(|\mathbf{x}-\mathbf{x}'|) + \int dtd\mathbf{x}\ \tilde{\mathbf{v}}.[-\partial_t\mathbf{v} - (\mathbf{v}.\boldsymbol{\nabla})\mathbf{v} + \nu_0\nabla^2\mathbf{v}]. \tag{4.6}$$

With the use of the complete weight functional

$$\mathcal{W} = \mathrm{e}^{S_1+S_2} \tag{4.7}$$

the expectation value of any desirable physical quantity is possible to be calculated.

Actions (4.3) and (4.6) for the studied model are written in the form convenient for the use of the standard Feynman diagrammatic technique. The inverse matrix of the quadratic



$$= \langle\psi\psi^\dagger\rangle_0 \equiv \Delta^{\psi\psi^\dagger}(\omega_k, \mathbf{k})$$

Figure 5: The propagators of the model

part of the actions determines the form of the bare propagators. It is easily seen that the

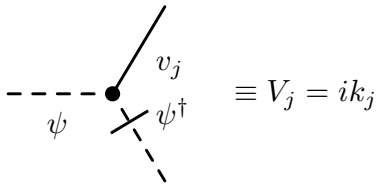


Figure 6: Interaction vertex describing advection

studied model contains three different types of propagators, together with two propagators depicted in Figure 2 we now have additional one depicted in Figure 5. In momentum-frequency representation it is given as

$$\Delta^{\psi\psi^\dagger}(\omega_k, \mathbf{k}) = \frac{1}{-i\omega_k + D_0 k^2} \tag{4.8}$$

and in the momentum-time representation as

$$\Delta^{\psi\psi^\dagger}(t, \mathbf{k}) = \theta(t)\exp(-D_0 k^2 t). \tag{4.9}$$

The advecting term from the action (4.3) can be similarly rewritten as was done in equation (2.24) for the vertex factor $\tilde{v}vv$

$$-\int dtd\mathbf{x}\ \psi^\dagger \boldsymbol{\nabla}(\mathbf{v}\psi) = \int dtd\mathbf{x}\ \psi^\dagger \partial_i(v_i\psi) = -\int dtd\mathbf{x}\ \psi^\dagger v_i \partial_i \psi = \int dtd\mathbf{x}\ (\partial_i \psi^\dagger) v_i \psi. \tag{4.10}$$

Rewriting this expression in the form $\psi^\dagger V_j v_j \psi$ we immediately obtain the vertex factor in the momentum space

$$V_j = ik_j, \tag{4.11}$$

where the momentum $\mathbf{k}$ represents the momentum flowing into the vertex through the field $\psi^\dagger$. Graphically this vertex is depicted in Figure 6. The two reaction vertices derived from the functional (4.3) according to the definition (2.23) are depicted in Figure 7 and physically describe the density fluctuations of the reactant particles. For completness let just note the interaction vertex $\tilde{v}vv$ responsible for interactions between velocity fluctuations is the same as in Figure 3.

It should be stressed that in our model there is no influence of the reactants on the velocity field itself. Therefore, the model given by actions (4.3) and (4.6) may be characterized as a model for the advection of the "passive" chemically active admixture.

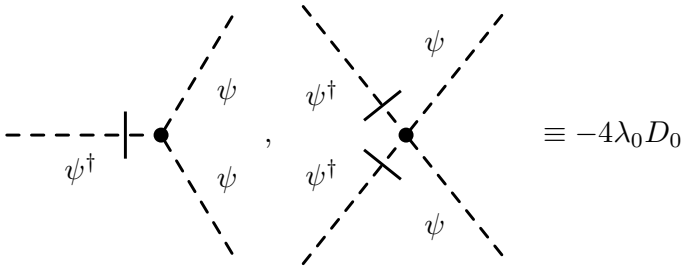


Figure 7: Interaction vertices responsible for density fluctuations and their corresponding vertex factor

4.3 UV RENORMALIZATION

The functional formulation provides a theoretical framework suitable for applying methods of quantum field theory. Using RG methods it is possible to determine the IR asymptotic (large spatial and time scales) behavior of the correlation functions. First of all a proper renormalization procedure is needed for the elimination of ultraviolet (UV) divergences. There are various renormalization prescriptions applicable for such a task, each with its own advantages. To most popular belong the Pauli-Villars, lattice and dimensional regularization [2]. In what follows we will employ the modified minimal subtraction ($\overline{\text{MS}}$) scheme. Strictly speaking, in the analytic renormalization there is no consistent MS scheme. What we mean here, is the ray scheme [66], in which the two regularizing parameters ϵ, Δ (ϵ has been introduced in (4.5) and $2\Delta = d-2$ was introduced in (3.5)) are taken proportional to each other: $\Delta = \xi\epsilon$, where the coefficient ξ is arbitrary but fixed. In this case, only one independent small parameter, say, ϵ remains and the notion of minimal subtraction becomes meaningful. UV divergences manifest themselves in the form of poles in the small expansion parameter and the minimal subtraction scheme is characterized by discarding all finite parts of the Feynman graphs in the calculation of the renormalization constants. In the modified scheme, as usual, certain geometric factors are not expanded in ϵ, however. This is the content of the $\overline{\text{MS}}$ scheme used in our analysis.

In order to apply the dimensional regularization for the evaluation of renormalization constants, an analysis of possible superficial divergences has to be performed. For the power counting in the actions (4.3) and (4.6) we use the scheme [31], in which to each quantity Q two canonical dimensions are assigned, one with respect to the wave number d_Q^k and the other to the frequency d_Q^ω. The normalization for these dimensions is

$$d_\omega^\omega = -d_t^\omega = 1, \quad d_k^k = -d_x^k = 1, \quad d_k^\omega = d_\omega^k = 0. \tag{4.12}$$

The canonical (engineering) dimensions for fields and parameters of the model are derived from the condition for action to be a scale-invariant quantity, i.e. to have a zero canonical dimension.

The quadratic part of the action (4.3) determines only the canonical dimension of the quadratic product $\psi^\dagger\psi$. In order to keep both terms in the nonlinear part of the action

$$\lambda_0 D_0 \int dt d\mathbf{x}[2\psi^\dagger + (\psi^\dagger)^2]\psi^2, \tag{4.13}$$

the field $\psi^\dagger$ must be dimensionless. If the field $\psi^\dagger$ has a positive canonical dimension, which is the case for $d > 2$, then the quartic term should be discarded as irrelevant by the power counting. The action with the cubic term only, however, does not generate any loop integrals corresponding to the density fluctuations and thus is uninteresting for the analysis of fluctuation effects in the space dimension $d = 2$.

Using the normalization choice (4.12) we are able to obtain the canonical dimensions for all the fields and parameters in the d-dimensional space. The results are summarized in Table 2. Here, $d_Q = d_Q^k + 2d_Q^\omega$ is the total canonical dimension and it is determined from

Q	ψ	$\psi^\dagger$	v	$\tilde{v}$	ν_0	D_0	λ_0	g_{10}	g_{20}
d_Q^k	d	0	-1	$d+1$	-2	-2	-2Δ	2ϵ	-2Δ
d_Q^ω	0	0	1	-1	1	1	0	0	0
d_Q	d	0	1	$d-1$	0	0	-2Δ	2ϵ	-2Δ

Table 2: Canonical dimensions for the parameters and the fields of the model

the condition that the parabolic differential operator of the diffusion and Navier-Stokes equation scale uniformly under the simultaneous momentum and frequency dilatation $k \to \mu k, \omega \to \mu^2\omega$.

The model is logarithmic when all coupling constants $g_{10}, g_{20}, \lambda_0$ vanish simultaneously. From Table 2 it follows that this situation occurs for the choice $\epsilon = \Delta = 0$. The UV divergences have the form of poles in various linear combinations of ϵ and Δ. The total canonical dimension of an arbitrary one-particle irreducible Green (1PI) function $\Gamma = \langle\Phi\ldots\Phi\rangle_{1-\mathrm{ir}}$ is given by the relation $d_\Gamma = d + 2 - N_\Phi d_\Phi$, where $N_\Phi = \{N_{\psi^\dagger}, N_\psi, N_v, N_{\tilde{v}}\}$ are the numbers of corresponding external fields [1]. The statistical averaging $\langle\ldots\rangle$ means averaging over all possible realizations of fields $\tilde{\mathbf{v}}, \mathbf{v}, \psi^\dagger, \psi$ satisfying appropriate boundary conditions with the use of the complete weight functional (4.7). Superficial UV divergences may be present only in those Γ functions for which d_Γ is a non-negative integer. The superficial

degree of divergence for a 1PI Green function Γ is

$$d_\Gamma = 4 - N_v - N_{\tilde{v}} - 2N_\psi. \tag{4.14}$$

However, the real degree of divergence δ_Γ is smaller, because of the structure of the interaction vertex (2.25),which allows for factoring out the differential operator ∂ to each external line $\tilde{v}$. The real divergence exponent δ_Γ may then be expressed as

$$\delta_\Gamma \equiv d_\Gamma - N_{\tilde{v}} = 4 - N_v - 2N_{\tilde{v}} - 2N_\psi \tag{4.15}$$

Although the canonical dimension for the field $\psi^\dagger$ is zero, there is no proliferation of superficial divergent graphs with arbitrary number of external $\psi^\dagger$ legs. This is due to the fact that $n_{\psi^\dagger} \leq n_\psi$, which may be established by a straightforward analysis of the graphs [24]. Brief analysis shows that the UV divergences are expected only for the 1PI Green functions listed in the Table 3. This theoretical analysis leads to the following

Γ_{1-ir}	$\langle\psi_\dagger\psi\rangle$	$\langle\psi_\dagger\psi v\rangle$	$\langle\tilde{v}v\rangle$	$\langle\tilde{v}vv\rangle$	$\langle\tilde{v}\tilde{v}\rangle$	$\langle\psi^\dagger\psi^2\rangle$	$\langle(\psi^\dagger)^2\psi^2\rangle$
d_Γ	2	1	2	1	2	0	0
δ_Γ	2	1	1	0	0	0	0

Table 3: Canonical dimensions for the (1PI) divergent Green functions of the model

renormalization of parameters g_0, D_0 and u_0

$$\begin{aligned} &g_1 = g_{10}\mu^{-2\epsilon}Z_1^3, \quad g_2 = g_{20}\mu^{2\Delta}Z_1^3Z_3^{-1}, \quad u = u_0Z_1Z_2^{-1}, \\ &\lambda = \lambda_0\mu^{2\Delta}Z_2Z_4^{-1}, \quad \nu = \nu_0Z_1^{-1}, \quad D = D_0Z_2^{-1}, \end{aligned} \tag{4.16}$$

where μ is the reference mass scale in the MS scheme [2] and we have introduced the inverse Prandtl number $u = D/\nu$ for convenience. It represents the ratio between diffusion and viscosity forces in a liquid. In terms of introduced renormalized parameters the total renormalized action for the annihilation reaction in a fluctuating velocity field is

$$\begin{aligned} S_R \;=\; &\int d\mathbf{x}dt\Big\{\psi^\dagger\partial_t\psi + \psi^\dagger\boldsymbol{\nabla}(\mathbf{v}\psi) - u\nu Z_2\nabla^2\psi + \lambda u\nu\mu^{-2\Delta}Z_4[2\psi^\dagger + (\psi^\dagger)^2]\psi^2 + \\ &n_0\int d\mathbf{x}\psi^\dagger(\mathbf{x},0) - \frac{1}{2}\tilde{\mathbf{v}}[g_1\nu^3\mu^{2\epsilon}(-\nabla^2)^{1-\Delta-\epsilon} - g_2\nu^3\mu^{-2\Delta}Z_3\nabla^2]\tilde{\mathbf{v}} + \\ &\tilde{\mathbf{v}}.[\partial_t\mathbf{v} + (\mathbf{v}.\boldsymbol{\nabla})\mathbf{v} - \nu Z_1\nabla^2\mathbf{v}]\Big\}. \end{aligned} \tag{4.17}$$

The renormalization constants $Z_i, i = 1,2,3,4$ are to be calculated perturbatively through the calculation of the UV divergent parts of the 1PI functions $\Gamma_{\psi^\dagger\psi}, \Gamma_{\psi^\dagger\psi^2}, \Gamma_{(\psi^\dagger)^2\psi^2}$, $\Gamma_{\tilde{v}v}$ and $\Gamma_{\tilde{v}\tilde{v}}$. Interaction terms corresponding to these functions have to be added to the original action $S = S_1 + S_2$ with the aim to ensure UV finiteness of all Green functions generated by the renormalized action S_R. At this stage the main goal is to calculate the renormalization constants $Z_i, i = 1,2,3,4$.

The singularities in various Green functions will be realized in the form of poles in ϵ and Δ and their linear combinations such as $2\epsilon + \Delta$ or $\epsilon - \Delta$. Recall that for the consistency of the $\overline{\text{MS}}$ scheme it is necessary that the ratio

$$\xi = \frac{\Delta}{\epsilon} \tag{4.18}$$

is a finite real number. It should be noted that the graphs corresponding to $\Gamma_{\psi^\dagger\psi^2}$ and $\Gamma_{(\psi^\dagger)^2\psi^2}$ differ only by one external vertex and thus give rise to equal renormalization of the rate constant $\lambda_0 D_0$. Therefore, in what follows, we will always consider the function $\Gamma_{\psi^\dagger\psi^2}$. In order to calculate the renormalization constants Z_2 and Z_4 we proceed according to the general scheme suggested in [66]. We require the fulfillment of UV finiteness (i.e.finite limit when $\epsilon, \Delta \to 0$) of the 1PI functions $\Gamma_{\psi^\dagger\psi}|_{\omega=0}$ and $\Gamma_{\psi^\dagger\psi^2}|_{\omega=0}$. Because the divergent part of the Feynman graphs should not depend on the value of ω, we have adopted the simplest choice $\omega = 0$. It is convenient to introduce the dimensionless expansion variables of the perturbation theory as

$$\alpha_{10} \equiv \frac{g_{10}\overline{S_d}}{p^{2\epsilon}}, \quad \alpha_{20} \equiv \frac{g_{20}\overline{S_d}}{p^{-2\Delta}}, \quad \alpha_{30} \equiv \frac{\lambda_0\overline{S_d}}{p^{-2\Delta}}, \tag{4.19}$$

where S_d is the surface area of the unit sphere in $d-$dimensional space, p is the total momentum flowing into the Feynman diagram and $\overline{S_d} = S_d/(2\pi)^d$. For brevity, in the following we use the abbreviation $\overline{g}_0 \equiv g_0\overline{S_d}$ for the parameters $\{g_{10}, g_{20}, \lambda_0\}$ or their renormalized counterparts, respectively. Next we demonstrate the perturbation series for the 1PI Green functions to the second order approximation. The perturbative expansion for $\Gamma_{\psi^\dagger\psi}$ may be written as

$$\Gamma_{\psi^\dagger\psi}|_{\omega=0} = D_0 p^2\Big[-1 + \sum_{\substack{n_1,n_2\geq 0,\\ n_1+n_2\geq 1}}^{n_1+n_2=2} \alpha_{10}^{n_1}\alpha_{20}^{n_2}\gamma_{\psi^\dagger\psi}^{(n_1,n_2)}(d,u_0)\Big], \tag{4.20}$$

where $\gamma_{\psi^\dagger\psi}$ are dimensionless coefficients which contain poles in ϵ and Δ. Explicit de-

pendence on the space dimension d and inverse Prandtl number u_0 is emphasized. It is important to note that there are no terms in this sequence proportional to the expansion parameter α_{30}. In terms of the renormalized parameters perturbative expansion for the Green function is (4.20)

$$\frac{\Gamma_{\psi^\dagger\psi}|_{\omega=0}}{Dp^2} = Z_2\Big[-1+\sum_{\substack{n_1,n_2\geq 0,\\ n_1+n_2\geq 1}}^{n_1+n_2=2}\alpha_1^{n_1}\alpha_2^{n_2}\gamma_{\psi^\dagger\psi}^{(n_1,n_2)}(d,u)\Big], \tag{4.21}$$

with the renormalized parameters $\alpha_1 = \overline{g_1}s^{2\epsilon}Z_1^{-3}$ and $\alpha_2 = \overline{g_2}s^{-2\Delta}Z_3Z_1^{-3}$ in accordance with the relations (4.16) and (4.19), where $s \equiv \mu/p$. Here we would like to stress, that in order to get the correct expansion in ϵ and Δ, one has to make replacement

$$d \to 2+2\Delta, \qquad u_0 \to Z_1^{-1}Z_2u, \tag{4.22}$$

in the arguments of $\gamma_{\psi^\dagger\psi}^{(n_1,n_2)}$. In the same way, the perturbation expansion series for the Green function $\Gamma_{\psi^\dagger\psi^2}$ is

$$\Gamma_{\psi^\dagger\psi^2}|_{\omega=0} = -4D_0\lambda_0\Big[1+\sum_{\substack{n_1,n_2,n_3\geq 0,\\ n_1+n_2+n_3\geq 1}}^{n_1+n_2+n_3=2}\alpha_{10}^{n_1}\alpha_{20}^{n_2}\alpha_{30}^{n_3}\gamma_{\psi^\dagger\psi^2}^{(n_1,n_2,n_3)}(d,u_0)\Big], \tag{4.23}$$

where $\gamma_{\psi^\dagger\psi^2}$ are dimensionless coefficients resulting from calculation of Feynman graphs. Again by replacing the bare parameters with the renormalized counterparts the series is obtained

$$\frac{\Gamma_{\psi^\dagger\psi^2}|_{\omega=0}}{4\lambda D\mu^{-2\Delta}} = -Z_4\Big[1+\sum_{\substack{n_1,n_2,n_3\geq 0,\\ n_1+n_2+n_3\geq 1}}^{n_1+n_2+n_3=2}\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}\gamma_{\psi^\dagger\psi^2}^{(n_1,n_2,n_3)}(d,u)\Big], \tag{4.24}$$

where the dimensionless parameter $\alpha_3 = \overline{\lambda}s^{-2\Delta}Z_2^{-1}Z_4$ is introduced and the change (4.22) is understood. Perturbation series for the Green function $\Gamma_{(\psi^\dagger)^2\psi^2}$ has the same form, so we do not present it.

Denoting by $Z^{(n)}$ the contribution of the order $g^n, g = \{g_1, g_2, \lambda\}$ the first order of renormalization constants Z_2 and Z_4 may be calculated via equations

$$Z_2^{(1)} = \mathscr{L}[\overline{g_1}s^{2\epsilon}\gamma_{\psi^\dagger\psi}^{(1,0)} + \overline{g_2}s^{-2\Delta}\gamma_{\psi^\dagger\psi}^{(0,1)}], \tag{4.25}$$

$$Z_4^{(1)} = -\mathscr{L}[\overline{g_1}s^{2\epsilon}\gamma_{\psi^\dagger\psi^2}^{(1,0,0)} + \overline{g_2}s^{-2\Delta}\gamma_{\psi^\dagger\psi^2}^{(0,1,0)} + \overline{\lambda}s^{-2\Delta}\gamma_{\psi^\dagger\psi^2}^{(0,0,1)}], \tag{4.26}$$

where $\mathscr{L}$ stands for the operation of extraction of the UV-divergent part (poles in ϵ and Δ or their linear combination). In the $\overline{\text{MS}}$ scheme finite terms are discarded, so we do not need to take care of them. At the second order the term for Z_2 can be schematically written as

$$\begin{aligned} Z_2^{(2)} = \mathscr{L}\Big[-\frac{\overline{g_1}s^{2\epsilon}}{1+u}\Big(uZ_2^{(1)} + (u+2)Z_1^{(1)}\Big)\gamma_{\psi^\dagger\psi}^{(1,0)} - \overline{g_2}s^{-2\Delta}\Big(\frac{u}{1+u}Z_2^{(1)} + \frac{u+2}{1+u}Z_1^{(1)} - \\ Z_3^{(1)}\Big)\gamma_{\psi^\dagger\psi}^{(0,1)} + \overline{g_1}^2 s^{4\epsilon}\gamma_{\psi^\dagger\psi}^{(2,0)} + \overline{g_1 g_2}s^{2\epsilon-2\Delta}\gamma_{\psi^\dagger\psi}^{(1,1)} + \overline{g_2}^2 s^{-4\Delta}\gamma_{\psi^\dagger\psi}^{(0,2)}\Big]. \end{aligned} \tag{4.27}$$

The two-loop graphs that contribute to the calculation of Z_2 are represented by the

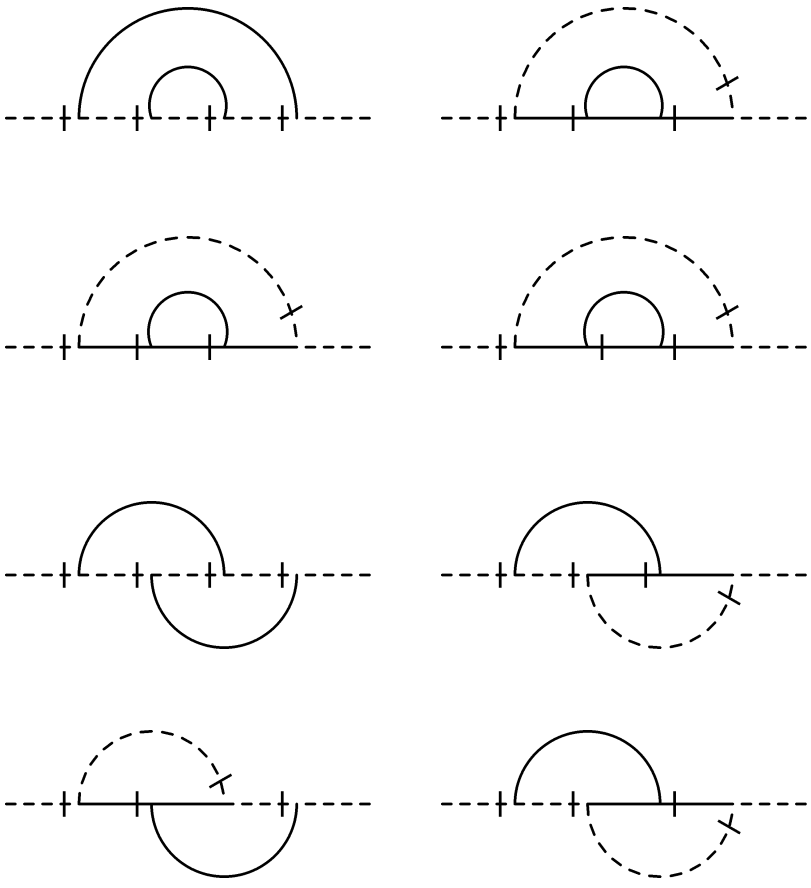


Figure 8: Two-loop graphs for the perturbation expansion of $\Gamma_{\psi^\dagger\psi}$

graphs depicted in Figure 8. For the renormalization constants Z_4 we have the expression

$$\begin{aligned} Z_4^{(2)} &= -\mathscr{L}[\overline{g_1}\overline{\lambda}s^{2(\epsilon-\Delta)}\gamma^{(1,0,1)}_{(\psi^+)^2\psi^2} + \overline{g_2}\overline{\lambda}s^{-4\Delta}\gamma^{(0,1,1)}_{\psi^\dagger\psi^2} + \overline{\lambda}^2 s^{-4\Delta}\gamma^{(0,0,2)}_{\psi^\dagger\psi^2} + \overline{g_1}s^{2\epsilon}\gamma^{(1,0,0)}_{\psi^\dagger\psi^2}(-3Z_1^{(1)}) + \\ &\quad \overline{g_2}s^{-2\Delta}\gamma^{(0,1,0)}_{\psi^\dagger\psi^2}(Z_3^{(1)} - 3Z_1^{(1)}) + \overline{\lambda}s^{-2\Delta}\gamma^{(0,0,1)}_{\psi^\dagger\psi^2}(2Z_4^{(1)} - Z_2^{(1)})]. \end{aligned} \tag{4.28}$$

The two-loop graphs that contribute to the calculation of Z_4 are represented by the

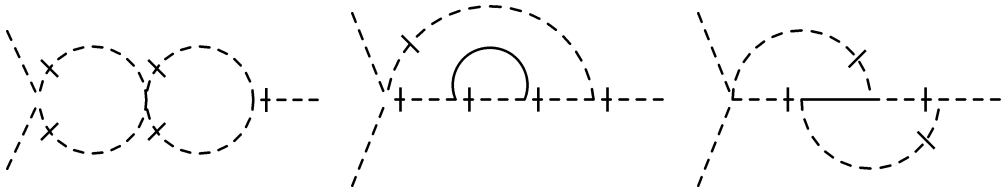


Figure 9: Two-loop graphs for the perturbation expansion of $\Gamma_{\psi^\dagger\psi^2}$

graphs depicted in Figure 9. Singular parts of all Feynman graphs needed for calculation of the renormalization constants Z_2 and Z_4 can be found in Appendix (A.1). From these expressions the renormalization constants Z_2 and Z_4 can be calculated in the form

$$\begin{aligned} Z_2 &= 1 - \frac{\overline{g_1}}{8u(1+u)\epsilon} + \frac{\overline{g_2}}{8u(1+u)\Delta} + \frac{A_{11}\overline{g_1}^2}{\epsilon^2} + \frac{A_{22}\overline{g_2}^2}{\Delta^2} + \frac{A_{12}\overline{g_1 g_2}}{\epsilon\Delta} + \\ &\quad \frac{B_{11}\overline{g_1}^2}{\epsilon} + \frac{B_{22}\overline{g_2}^2}{\Delta} + \frac{B_{12}\overline{g_1 g_2}}{\epsilon-\Delta}, \end{aligned} \tag{4.29}$$

$$\begin{aligned} Z_4 &= 1 - \frac{\overline{\lambda}}{2\Delta} - \frac{1}{16u(1+u)}\frac{\overline{g_1\lambda}}{(\epsilon-\Delta)\Delta} + \frac{1}{32u(1+u)}\frac{\overline{g_2\lambda}}{\Delta^2} + \frac{\overline{\lambda}^2}{4\Delta^2} - \\ &\quad \Big(\frac{\overline{g_1\lambda}}{\epsilon-\Delta} - \frac{\overline{g_2\lambda}}{\Delta}\Big)C(u,\xi). \end{aligned} \tag{4.30}$$

The lengthy expressions for the coefficient functions $A_{ij}(\xi,u)$,$B_{ij}(\xi,u)$ and $C(u,\xi)$ can be found in the (A.2).

In a similar way we obtain renormalization constants Z_1 and Z_3 [66] from condition of the UV finiteness for the 1PI Green functions $\Gamma_{\tilde{v}v}|_{\omega=0}$ and $\Gamma_{\tilde{v}\tilde{v}}|_{\omega=0}$. The perturbation series for $\Gamma_{\tilde{v}v}$ can be written as

$$\Gamma_{\tilde{v}v}|_{\omega=0} = \nu_0 p^2 P^p_{ij}\Big[-1 + \sum_{\substack{n_1,n_2\geq 0,\\ n_1+n_2\geq 1}}^{n_1+n_2=2} \alpha_{10}^{n_1}\alpha_{20}^{n_2}\gamma^{(n_1,n_2)}_{\tilde{v}v}(d)\Big], \tag{4.31}$$

and for $\Gamma_{\tilde{v}\tilde{v}}$ as

$$\Gamma_{\tilde{v}\tilde{v}}|_{\omega=0} = P^p_{ij}\Big[g_{10}\nu_0^3 p^{2-2\Delta-2\epsilon} + g_{20}\nu_0^3 p^2\Big\{-1 + \sum_{\substack{n_1\geq 0, n_2\geq -1,\\ n_1+n_2\geq 1}}^{n_1+n_2=2} \alpha_{10}^{n_1}\alpha_{20}^{n_2}\gamma_{\tilde{v}\tilde{v}}^{(n_1,n_2)}(d)\Big\}\Big]. \quad (4.32)$$

From the definition of the projection operator P^p_{ij} it is easy to see, that after contracting indices i and j we are left with the constant $d-1$. Hence, rewriting perturbations series for $\Gamma_{\tilde{v}v}$ and $\Gamma_{\tilde{v}\tilde{v}}$ in the renormalized variables (4.16) and contracting indices i and j we get

$$\frac{\Gamma_{\tilde{v}v}|_{\omega=0}}{\nu p^2(d-1)} = -Z_1 + Z_1\Big[\sum_{\substack{n_1,n_2\geq 0,\\ n_1+n_2\geq 1}}^{n_1+n_2=2} \alpha_1^{n_1}\alpha_2^{n_2}\gamma_{\tilde{v}v}^{(n_1,n_2)}(d)\Big], \quad (4.33)$$

$$\frac{\Gamma_{\tilde{v}\tilde{v}}|_{\omega=0}}{(d-1)g_2\nu^3\mu^{-2\Delta}p^2} = \frac{g_1}{g_2}s^{2\epsilon+2\Delta} + Z_3 + Z_3\times\sum_{\substack{n_1\geq 0, n_2\geq -1,\\ n_1+n_2\geq 1}}^{n_1+n_2=2} \alpha_1^{n_1}\alpha_2^{n_2}\gamma_{\tilde{v}\tilde{v}}^{(n_1,n_2)}(d). \quad (4.34)$$

By the same algorithm as described above in detail for the calculation Z_2 and Z_4 explicit expressions for the renormalization constants Z_1 and Z_3 are obtained. The results for them in the $\overline{\text{MS}}$ scheme can be found in [66].

4.4 IR STABLE FIXED POINTS AND SCALING REGIMES

The coefficient functions of the RG differential operator for the Green functions

$$D_{\text{RG}} = \mu\frac{\partial}{\partial\mu}\Big|_0 = \mu\frac{\partial}{\partial\mu} + \sum_{g_i}\beta_i\frac{\partial}{\partial g_i} - \gamma_1\nu\frac{\partial}{\partial\nu}, \quad (4.35)$$

where the bare parameters are denoted with the subscript “0”, are defined as

$$\gamma_1 = \mu\frac{\partial \ln Z_1}{\partial\mu}\Big|_0, \qquad \beta_i = \mu\frac{\partial g_i}{\partial\mu}\Big|_0, \quad (4.36)$$

with the charges $g_i = \{g_1, g_2, u, \lambda\}$. From this definition and the renormalization relations

(4.16) it follows that

$$\beta_{g_1} = g_1(-2\epsilon + 3\gamma_1), \qquad \beta_{g_2} = g_2(2\Delta + 3\gamma_1 - \gamma_3),$$
$$\beta_\lambda = \lambda(2\Delta - \gamma_4 + \gamma_2), \quad \beta_u = u(\gamma_1 - \gamma_2), \tag{4.37}$$

where the anomalous dimensions γ_α ($\alpha = 2,3,4$) are defined as

$$\gamma_\alpha = \mu \frac{\partial \ln Z_\alpha}{\partial \mu}\Big|_0. \tag{4.38}$$

We are interested in the IR asymptotics of small momentum $\mathbf{p}$ and frequencies ω of the renormalized functions or, equivalently, large relative distances and time differences in the $(t,\mathbf{x})$ representation. Such a behavior is governed by the IR-stable fixed point $g^* = (g_1^*, g_2^*, u^*, \lambda^*)$, which are determined as zeroes of the β functions $\beta(g^*) = 0$. The fixed point g^* is IR stable, if real parts of all eigenvalues of the matrix $\omega_{ij} \equiv \partial\beta_i/\partial g_j|_{g=g^*}$ are strictly positive. From the knowledge of renormalization constants Z_2 and Z_4 (4.29),(4.30) and definitions (4.36),(4.38) it is possible to calculate anomalous dimensions γ_2 and γ_4

$$\gamma_2 = \frac{\overline{g_1} + \overline{g_2}}{4u(1+u)} - 4B_{11}\overline{g_1}^2 + 4B_{22}\overline{g_2}^2 - 2B_{12}\overline{g_1 g_2}, \tag{4.39}$$
$$\gamma_4 = -\overline{\lambda} + \overline{\lambda}(\overline{g_1} + \overline{g_2})C(u,\xi). \tag{4.40}$$

A straightforward calculation shows that higher order poles cancel each other, so that the anomalous dimensions γ_2 and γ_4 are finite. For completeness we quote also anomalous dimensions γ_1 and γ_3 [66] to the same order

$$\gamma_1 = \frac{\overline{g_1} + \overline{g_2}}{16} + \frac{(4\xi+3)}{512(2+\xi)}\overline{g_1}^2 + \frac{5\xi+3}{512}\overline{g_1 g_2} - \frac{R}{256}(\overline{g_1} + \overline{g_2})^2, \tag{4.41}$$

$$\gamma_3 = \frac{(\overline{g_1} + \overline{g_2})^2}{16\overline{g_2}} - \frac{\xi(13+19\xi)}{1024(2+\xi)}\frac{\overline{g_1}^3}{\overline{g_2}} + \frac{34\xi + 19 + 6\xi^2}{512(2+\xi)}\overline{g_1}^2 - \frac{3\overline{g_1}^2}{512} + \frac{13+31\xi}{1024}\overline{g_1 g_2} + \frac{1-R}{256}\frac{(\overline{g_1} + \overline{g_2})^3}{\overline{g_2}}, \tag{4.42}$$

where the value $R = -0.168$ is a result from numerical integration. Zeroes of the beta functions (4.37) determine possible IR behavior of the model. There are four IR stable fixed points and one IR unstable fixed point. In this section we present them with their regions of stability.

(*i*) The trivial (Gaussian) fixed point

$$\overline{g_1}^* = \overline{g_2}^* = \overline{\lambda}^* = 0, \tag{4.43}$$

with no restrictions on the inverse Prandtl number u. The Gaussian fixed point is stable, when

$$\epsilon < 0, \qquad \Delta > 0. \tag{4.44}$$

and physically corresponds to the case, when the mean-field solution is valid and fluctuation effects negligible.

(*ii*) The short-range (thermal) fixed point

$$\begin{aligned}
&\overline{g_1}^* = 0, \quad \overline{g_2}^* = -16\Delta + 8(1+2R)\Delta^2, \\
&u^* = \frac{\sqrt{17}-1}{2} - 1.12146\Delta, \\
&\overline{\lambda}^* = -\Delta + \frac{\Delta^2}{2}(\xi - 2.64375),
\end{aligned} \tag{4.45}$$

at which local correlations of the random force dominate over the long-range correlations. This fixed point has the following basin of attraction

$$\Delta - \frac{2R-1}{2}\Delta^2 < 0, \quad 2\epsilon + 3\Delta - \frac{3\Delta^2}{2} < 0, \tag{4.46}$$

$$\Delta + \frac{1}{2}\Delta^2 < 0, \quad \Delta + 0.4529\Delta\epsilon < 0 \tag{4.47}$$

and corresponds to anomalous decay faster than that due to density fluctuations only, but slower than the mean-field decay.

(*iii*) The kinetic [50] fixed point with finite rate coefficient

$$\begin{aligned}
&\overline{g_1}^* = \frac{32}{9}\frac{\epsilon(2\epsilon+3\Delta)}{\epsilon+\Delta} + g_{12}^*(\xi)\epsilon^2, \\
&\overline{g_2}^* = \frac{32}{9}\frac{\epsilon^2}{\Delta+\epsilon} + g_{22}^*(\xi)\epsilon^2, \\
&u^* = \frac{\sqrt{17}-1}{2} + u_1^*(\xi)\epsilon, \\
&\overline{\lambda}^* = -\frac{2}{3}(\epsilon+3\Delta) + \frac{1}{9\pi}(3\Delta+\epsilon)(Q\epsilon-\Delta),
\end{aligned} \tag{4.48}$$

Here $Q = 1.64375$. The fixed point (4.48) is stable, when inequalities

$$\Omega_{\pm} > 0\,, \qquad \epsilon > 0\,, \qquad -\frac{2}{3}\epsilon < \Delta < -\frac{1}{3}\epsilon, \tag{4.49}$$

are fulfilled, where

$$\begin{aligned}\Omega_{\pm} \;=\; & \Delta + \frac{4}{3}\epsilon \pm \frac{\sqrt{9\Delta^2 - 12\epsilon\Delta - 8\epsilon^2}}{3} + \frac{2}{9}\Big(-(3+2R)\epsilon^2 - 3\epsilon\Delta \pm \\ & \frac{4\epsilon(\epsilon+3\Delta)R - 6\epsilon^2 - 12\epsilon\Delta - 9\Delta^2}{\sqrt{9\Delta^2 - 12\Delta\epsilon - 8\epsilon^2}}\epsilon\Big)\end{aligned} \tag{4.50}$$

The decay rate controlled by this fixed point of the average number density is faster than the decay rate induced by dominant local force correlations, but still slower than the mean-field decay rate.

(*iv*) The kinetic fixed point with vanishing rate coefficient:

$$\begin{aligned}\overline{g_1}^* &= \frac{32}{9}\frac{\epsilon(2\epsilon+3\Delta)}{\epsilon+\Delta} + g_{12}^*(\xi)\epsilon^2, \\ \overline{g_2}^* &= \frac{32}{9}\frac{\epsilon^2}{\Delta+\epsilon} + g_{22}^*(\xi)\epsilon^2, \\ u^* &= \frac{\sqrt{17}-1}{2} + u_1^*(\xi)\epsilon\,, \quad \overline{\lambda}^* = 0\,.\end{aligned} \tag{4.51}$$

This fixed point is stable, when the long-range correlations of the random force are dominant

$$\Omega_{\pm} > 0\,, \qquad \epsilon > 0\,, \qquad \Delta > -\frac{1}{3}\epsilon\,, \tag{4.52}$$

and corresponds to reaction kinetics with the normal (mean-field like) decay rate.

(*v*) Driftless fixed point given by

$$\overline{g_1}^* = \overline{g_2}^* = 0, \quad u^* \text{ not fixed}, \quad \overline{\lambda}^* = -2\Delta, \tag{4.53}$$

with the following eigenvalues

$$\Omega_1 = -2\epsilon, \quad \Omega_2 = -\Omega_4 = 2\Delta, \quad \Omega_3 = 0. \tag{4.54}$$

An analysis of the structure of the fixed points and the basins of attraction leads to the following physical picture of the effect of the random stirring on the reaction kinetics. Anomalous behavior always emerges below two dimensions, when the *local* correlations are dominant in the spectrum of the random forcing [the short-range fixed point

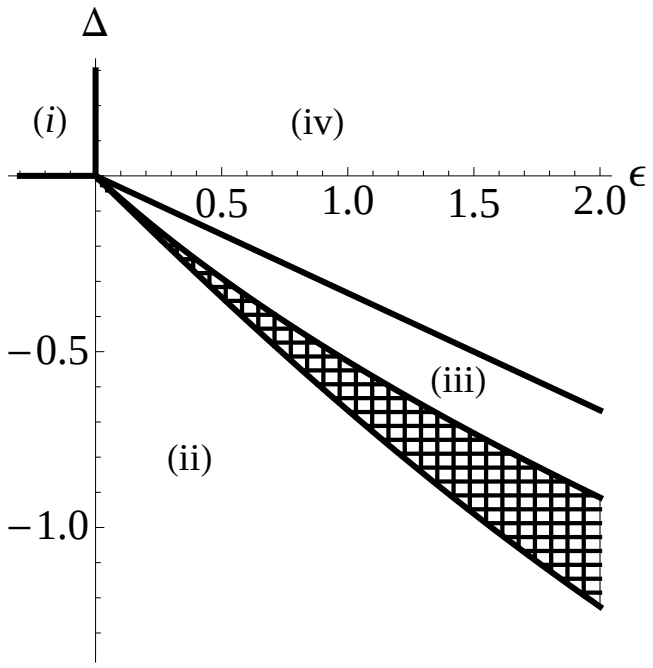
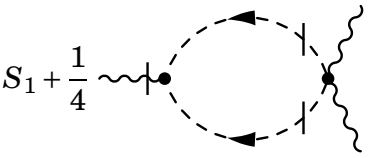
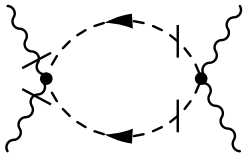


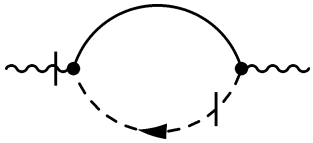
Figure 10: Regions of stability

(*ii*)]. However, the random stirring gives rise to an effective reaction rate faster than the density-fluctuation induced reaction rate even in this case. The anomaly is present (but with still faster decay, see the next Section) also, when the long-range part of the forcing spectrum is effective, but the powerlike falloff of the correlations is fast [this regime is governed by the kinetic fixed point (*iii*)]. Note that this is different from the case in which the divergenceless random velocity field is time-independent, in which case there is no fixed point with $\lambda^* \neq 0$[63]. At slower spatial falloff of correlations, however, the anomalous reaction kinetics is replaced by a mean-field-like behavior [this corresponds to the kinetic fixed point (*iv*)]. In particular, in dimensions $d > 1$ this is the situation for the value $\epsilon = 2$ which corresponds to the Kolmogorov spectrum of the velocity field in fully developed turbulence. Thus, long-range correlated forcing gives rise to a random velocity field, which tends to suppress the effect of density fluctuations on the reaction kinetics below two dimensions. For better illustration, regions of stability for fixed points (*i*)-(*iv*) are depicted in Fig.10. Wee see that in contrast to the one-loop approximation [51], overlap (dashed region) between regions of stability of fixed points (*ii*) and (*iii*) is observed. It is a common situation in the perturbative RG approach that higher order terms lead to either gap or overlap between neighbouring stability regions. The physical realization of the large-scale behavior then depends on the initial state of the system.

4.5 LONG-TIME ASYMPTOTICS OF NUMBER DENSITY

Since the renormalization and calculation of the fixed points of the RG are carried out at two-loop level, we are able to find the first two terms of the ϵ, Δ expansion of the average number density, which corresponds to solving the stationarity equations at the one-loop level. The simplest way to find the average number density is to calculate it from the stationarity condition of the functional Legendre transform [32] (which is often called the effective action) of the generating functional obtained by replacing the unrenormalized action by the renormalized one in the weight functional. This is a convenient way to avoid any summing procedures used [24] to take into account the higher-order terms in the initial number density n_0. We are interested in the solution for the number density, therefore we put the expectation values of the fields $\mathbf{v}$ and $\tilde{\mathbf{v}}$ equal to zero at the outset (but retain, of course, the propagator and the correlation function). Therefore, at the second-order approximation the effective renormalized action for this model is

$\Gamma_R = S_1 + \frac{1}{4}$  $+ \frac{1}{8}$  $+$

 $+ \ldots ,$ (4.55)

where S_1 is the action (4.3) (within our convention $S_2 = 0$ in the effective action) and graphs are shown together with their symmetry coefficients. The slashed wavy line corresponds to the field $\psi^\dagger$ and the single wavy line to the field ψ. The stationarity equations for the variational functional

$$\frac{\delta\Gamma_R}{\delta\psi^\dagger} = \frac{\delta\Gamma_R}{\delta\psi} = 0 \tag{4.56}$$

give rise to the equations

$$\begin{aligned}
\partial_t\psi &= uvZ_2\nabla^2\psi - 2\lambda uv\mu^{-2\Delta}Z_4\left(1+\psi^\dagger\right)\psi^2 \\
&\quad +4u^2v^2\lambda^2\mu^{-4\Delta}\int_0^\infty dt'\int dy\,(\Delta^{\psi\psi^\dagger})^2(t-t',\boldsymbol{x}-\boldsymbol{y})\psi^2(t',\boldsymbol{y}) \\
&\quad +4u^2v^2\lambda^2\mu^{-4\Delta}\psi^\dagger(t,\boldsymbol{x})\int_0^\infty dt'\int d\boldsymbol{y}\,(\Delta^{\psi\psi^\dagger})^2(t-t',\boldsymbol{x}-\boldsymbol{y})\psi^2(t',\boldsymbol{y}) \\
&\quad +\frac{\partial}{\partial x_i}\int_0^\infty dt'\int dy\,\Delta_{ij}^{vv}(t-t',\boldsymbol{x}-\boldsymbol{y})\frac{\partial}{\partial x_j}\Delta^{\psi\psi^\dagger}(t-t',\boldsymbol{x}-\boldsymbol{y})\psi(t',\boldsymbol{y})+\ldots, \qquad (4.57)
\end{aligned}$$

$$\begin{aligned}
-\partial_t\psi^\dagger &= uvZ_2\nabla^2\psi^\dagger - 2\lambda uv\mu^{-2\Delta}Z_4\left[2\psi^\dagger+\left(\psi^\dagger\right)^2\right]\psi \\
&\quad +8u^2v^2\lambda^2\mu^{-4\Delta}\int_0^\infty dt'\int d\boldsymbol{y}(\Delta^{\psi\psi^\dagger})^2(t'-t,\boldsymbol{y}-\boldsymbol{x})\psi^\dagger(t',\boldsymbol{y})\psi(t,\boldsymbol{x}) \\
&\quad +4u^2v^2\lambda^2\mu^{-4\Delta}\int_0^\infty dt'\int d\boldsymbol{y}(\Delta^{\psi\psi^\dagger})^2(t'-t,\boldsymbol{y}-\boldsymbol{x})\times\left[\psi^\dagger(t',\boldsymbol{y})\right]^2\psi(t,\boldsymbol{x})+ \\
&\quad \int_0^\infty dt'\int dy\,\Delta_{ji}^{vv}(t'-t,\boldsymbol{y}-\boldsymbol{x})\frac{\partial}{\partial x_i}\Delta^{\psi\psi^\dagger}(t'-t,\boldsymbol{y}-\boldsymbol{x})\frac{\partial}{\partial y_j}\psi^\dagger(t',\boldsymbol{y})+\ldots \qquad (4.58)
\end{aligned}$$

In (4.57) and (4.58), in the integral terms it is sufficient to put all renormalization constants equal to unity. Substituting the solution $\psi^\dagger = 0$ of (4.58) into (4.57) we arrive at the fluctuation-amended rate equation in the form

$$\begin{aligned}
\partial_t\psi &= uvZ_2\nabla^2\psi - 2\lambda uv\mu^{-2\Delta}Z_4\psi^2 \qquad (4.59) \\
&\quad +4u^2v^2\mu^{-4\Delta}\lambda^2\int_0^\infty dt'\int dy\,(\Delta^{\psi\psi^\dagger})^2(t-t',\boldsymbol{x}-\boldsymbol{y})\psi^2(t',\boldsymbol{y}) \\
&\quad +\frac{\partial}{\partial x_i}\int_0^\infty dt'\int dy\,\Delta_{ij}^{vv}(t-t',\boldsymbol{x}-\boldsymbol{y})\frac{\partial}{\partial x_j}\Delta^{\psi\psi^\dagger}(t-t',\boldsymbol{x}-\boldsymbol{y})\psi(t',\boldsymbol{y})+\ldots, \qquad (4.60)
\end{aligned}$$

This is a nonlinear partial integro-differential equation, whose explicit solution is not known. It is readily seen that for a homogeneous solution the term resulting from the third graph in (4.55) vanishes and hence the influence of the velocity field on the homogeneous annihilation process would be only through the renormalization of the coefficients

λ and D. However, in case of a nonuniform density field ψ the effect of velocity fluctuations is explicit in (4.59). Such a solution can be most probably found only numerically.

To arrive at an analytic solution, we restrict ourselves to the homogeneous density $n(t) = \langle\psi(t)\rangle$, which can be identified with the expression (3.61). In this case the last term in (4.59) vanishes together with the Laplace operator term and the remaining coordinate integral may be calculated explicitly . The propagator is the diffusion kernel of the renormalized model (we consider first the system in the general space dimension d)

$$\Delta^{\psi\psi^\dagger}(t-t',\boldsymbol{x}) = \frac{\theta(t-t')}{[4\pi u\nu(t-t')]^{d/2}} \exp\left[-\frac{x^2}{4u\nu(t-t')}\right]. \tag{4.61}$$

As noted above, for calculation of the one-loop contribution it is sufficient to put the renormalization constant $Z_2 = 1$ in the propagator $\Delta^{\psi\psi^\dagger}$. Therefore, evaluation of the Gaussian coordinate integral in (4.59) yields

$$\int d\boldsymbol{y}(\Delta^{\psi\psi^\dagger})^2(t-t',\boldsymbol{x}-\boldsymbol{y}) = \frac{\theta(t-t')}{[8\pi u\nu(t-t')]^{d/2}} \tag{4.62}$$

and we arrive at the ordinary integro-differential equation

$$\frac{\mathrm{d}n(t)}{\mathrm{d}t} = -2\lambda u\nu\mu^{-2\Delta}Z_4 n^2(t) + 4\lambda^2u^2\nu^2\mu^{-4\Delta}\int_0^t dt' \frac{n^2(t')}{[8\pi u\nu(t-t')]^{d/2}}. \tag{4.63}$$

Spatial fluctuations in the particle density show in the integral term and affect rather heavily even the homogeneous solution. In particular, the integral in (4.63) diverges at the upper limit in space dimensions $d \geq 2$. This is a consequence of the UV divergences in the model above the critical dimension $d_c = 2$ and near the critical dimension is remedied by the UV renormalization of the model. To see this, subtract and add the term $n^2(t)$ in the integrand to obtain

$$\begin{aligned}\frac{\mathrm{d}n(t)}{\mathrm{d}t} &= -2\lambda u\nu\mu^{-2\Delta}Z_4 n^2(t) + 4\lambda^2u^2\nu^2\mu^{-4\Delta}n^2(t)\int_0^t \frac{dt'}{[8\pi u\nu(t-t')]^{d/2}} \\ &\quad + 4\lambda^2u^2\nu^2\mu^{-4\Delta}\int_0^t dt' \frac{n^2(t')-n^2(t)}{[8\pi u\nu(t-t')]^{d/2}}.\end{aligned} \tag{4.64}$$

The last integral here is now convergent at least near two dimensions, provided the solution $n(t)$ is a continuous function. This is definitely the case for the iterative solution

constructed below. The divergence in the first integral in (4.64) may be explicitly calculated below two dimensions and is canceled – in the leading order in the parameter $\Delta = (d-2)/2$ – by the one-loop term of the renormalization constant Z_4 (4.30). Expanding the right-hand side of (4.64) in the parameter $\Delta = (d-2)/2$ to the next-to-leading order we arrive at the equation

$$\begin{aligned}\frac{\mathrm{d}n(t)}{\mathrm{d}t} &= -2\lambda u v \mu^{-2\Delta} n^2(t) + 2\lambda u v \mu^{-2\Delta} n^2(t) \left\{ \frac{\lambda}{4\pi} \left[\gamma + \ln\left(2uv\mu^2 t\right) \right] \right\} \\ &\quad + \frac{\lambda^2 u v \mu^{-2\Delta}}{2\pi} \int_0^t dt' \, \frac{n^2(t') - n^2(t)}{t - t'} . \end{aligned} \tag{4.65}$$

without divergences near two dimensions. Here, the factor $\mu^{-2\Delta}$ has been retained intact in order not to spoil the consistency of scaling dimensions in different terms of the equation. In (4.65), $\gamma = 0.57721$ is Euler's constant and we have considered the coupling constant λ and the parameter $\Delta = (d-2)/2$ to be small parameters of the same order taking into account the magnitudes of the parameters in the basins of attraction of the fixed points of the RG. The leading-order approximation for $n(t)$ is given by the first term on the right-hand side of (4.65) and it is readily seen that after substitution of this expression the integral term in (4.65) is of the order of λ^3 and thus negligible in the present next-to-leading-order calculation. In this approximation, Equation (4.65) yields

$$n(t) = \frac{n_0}{1 + 2\lambda u v t \left\{ 1 + \frac{\lambda}{4\pi} \left[1 - \gamma - \ln\left(2uv\mu^2 t\right) \right] \right\} \mu^{-2\Delta} n_0} , \tag{4.66}$$

where n_0 is the initial number density.

Since the fields $\Phi = \{v, \tilde{v}, \psi, \psi^\dagger\}$ are not renormalized, the renormalized connected Green functions W_R differ from the unrenormalized $W = \langle \Phi \ldots \Phi \rangle$ [1] only by the choice of parameters and thus one may write

$$W_R(g, v, \mu, \ldots) = W(g_0, v_0, \ldots), \tag{4.67}$$

where $g_0 = \{g_{10}, g_{20}, u_0, \lambda_0\}$ is the full set of the bare parameters and dots denotes all variables unaffected by the renormalization procedure. The independence of renormalization mass parameter μ is expressed by the equation $\mu \partial_\mu W_R = 0$. Using this equation

the RG equation for the mean particle number $n(t)$ is readily obtained:

$$\left(\mu\frac{\partial}{\partial\mu}+\sum_g\beta_g\frac{\partial}{\partial g}-\gamma_1\nu\frac{\partial}{\partial\nu}\right)n(t,\mu,\nu,n_0,g)=0. \tag{4.68}$$

We are interested in long-time behavior of the system ($t\to\infty$), therefore we trade the renormalization mass for the time variable. Canonical scale invariance yields relations [36]

$$\left(\mu\frac{\partial}{\partial\mu}-2\nu\frac{\partial}{\partial\nu}+dn_0\frac{\partial}{\partial n_0}-d\right)n(t,\mu,\nu,n_0,g)=0, \tag{4.69}$$

$$\left(-t\frac{\partial}{\partial t}+\nu\frac{\partial}{\partial\nu}\right)n(t,\mu,\nu,n_0,g)=0, \tag{4.70}$$

where the first equation expresses scale invariance with respect to wave number and the second equation with respect to time. Eliminating partial derivatives with respect to the renormalization mass μ and viscosity ν we obtain the Callan-Symanzik equation for the mean particle number

$$\left[(2-\gamma_1)t\frac{\partial}{\partial t}+\sum_g\beta_g\frac{\partial}{\partial g}-dn_0\frac{\partial}{\partial n_0}+d\right]n\left(t,\mu,\nu,n_0,g\right)=0 \tag{4.71}$$

To separate information given by the RG, consider the dimensionless normalized mean particle number

$$\frac{n}{n_0}=\Phi\left(\nu\mu^2t,\lambda u\frac{n_0}{\mu^d},g\right). \tag{4.72}$$

For the asymptotic analysis, it is convenient to express the particle density in the combination used here. Solution of (4.71) by the method of characteristics yields

$$\Phi\left(\nu\mu^2t,\lambda u\frac{n_0}{\mu^d},g\right)=\Phi\left(\nu\mu^2\tau,\overline{\lambda}\overline{u}\frac{\overline{n}_0}{\mu^d},\overline{g}\right) \tag{4.73}$$

where τ is the time scale. In Equation (4.73), $\overline{g}$ and $\overline{n}_0$ are the first integrals of the system of differential equations

$$t\frac{d}{dt}\overline{g}=-\frac{\beta_g(\overline{g})}{2-\gamma_1(\overline{g})},\quad t\frac{d}{dt}\overline{n}_0=d\frac{\overline{n}_0}{2-\gamma_1(\overline{g})}. \tag{4.74}$$

Here $\overline{g}=\{\overline{g_1},\overline{g_2},\overline{u},\overline{\lambda}\}$ with initial conditions $\overline{g}|_{t=\tau}=g$ and $\overline{n}_0|_{t=\tau}=n_0$. In particular,

$$\overline{\lambda}\overline{u}\overline{n_0}=\lambda u\,n_0\left(\frac{t}{\tau}\right)\exp\left[\int_\tau^t\frac{\gamma_4ds}{(2-\gamma_1)s}\right]. \tag{4.75}$$

The asymptotic expression of the integral on the right-hand side of (4.75) in the vicinity of the IR-stable fixed point g^* is of the form

$$\int_\tau^t \frac{\gamma_4 ds}{(2-\gamma_1)s} \underset{t\to\infty}{\sim} \frac{\gamma_4^*}{2-\gamma_1^*}\ln\left(\frac{t}{\tau}\right) + \frac{2}{2-\gamma_1^*}\int_\tau^\infty \frac{(\gamma_4-\gamma_4^*)ds}{(2-\gamma_1)s} = \frac{\gamma_4^*}{2-\gamma_1^*}\ln\left(\frac{t}{\tau}\right) + \tilde{c}_4(\tau), \tag{4.76}$$

corrections to which vanish in the limit $t\to\infty$. In (4.76) and henceforth, the notation $\gamma_1^* = \gamma_1(g^*)$ has been used. From the point of view of the long-time asymptotic behavior the next-to-leading term in (4.76) is an inessential constant. In the vicinity of the fixed point

$$\overline{\lambda}\overline{u}\frac{\overline{n}_0}{\mu^d} \sim \lambda u \frac{n_0}{\mu^d}\left(\frac{t}{\tau}\right)^{1+\frac{\gamma_4^*}{2-\gamma_1^*}} \widetilde{C}_n \equiv \lambda u \frac{n_0}{\mu^d}\left(\frac{t}{\tau}\right)^{\alpha} \widetilde{C}_n \equiv \overline{y}\widetilde{C}_n\,, \tag{4.77}$$

where a shorthand notation $\overline{y}$ has been introduced for the long-time scaling of the normalized number density as well as the dimensional normalization constant

$$\widetilde{C}_n = e^{\tilde{c}_4(\tau)}.$$

and the decay exponent

$$\alpha = 1 + \frac{\gamma_4^*}{2-\gamma_1^*} \tag{4.78}$$

The asymptotic behavior of the normalized particle density is described by the scaling function $f(x,y)$

$$\Phi\left(\nu\mu^2 t, \lambda u\frac{n_0}{\mu^d}, g\right) \sim \Phi\left(\nu\mu^2\tau, \widetilde{C}_n\overline{y}, g^*\right) \equiv f\left(\nu\mu^2\tau, \widetilde{C}_n\overline{y}\right). \tag{4.79}$$

The free parameters in the variables of the scaling function $f(x,y)$ correspond to the choice of units of these variables, whereas the objective information is contained in the form of the scaling function [36, 1]. Here, it is convenient to use the explicit solution (4.66) to obtain the ε, Δ expansion for the inverse $h(x,y) = 1/f(x,y)$ of the scaling function. We obtain the generic expression

$$h(x,y) = \frac{1}{f(x,y)} = 1 + 2xy\left\{1 + \frac{\lambda^*}{4\pi}\left[1-\gamma-\ln\left(2u^*x\right)\right]\right\}, \tag{4.80}$$

the substitution in which of the various fixed-point values λ^* (at the leading order $\lambda^* \approx 2\pi\overline{\lambda}^*$) and u^* in the leading approximation yields the corresponding ε, Δ expansions.

Below, we list the scaling functions $h(x,y)$ and the dynamic exponents α at the stable

fixed points in the next-to-leading-order approximation.
(*i*) At the trivial (Gaussian) fixed point (4.43) the mean-field behavior takes place with

$$h(x,y) = 1 + 2xy\,,$$
$$\alpha = 1\,. \tag{4.81}$$

(*ii*) The thermal (short-range) fixed point (4.45) leads to scaling function and decay exponent

$$h(x,y) = 1 + 2xy\left\{1 - \frac{\Delta}{2}\left[1 - \gamma - \ln\left(\sqrt{17} - 1\right)x\right]\right\},$$
$$\alpha = 1 + \frac{\Delta}{2} + \frac{\Delta^2}{2}\,. \tag{4.82}$$

Here, the last coefficient is actually a result of numerical calculation, which in the standard accuracy of Mathematica is equal to 0.5. We have not been able to sort out this result analytically, but think that most probably the coefficient of the Δ^2 term in the decay exponent α in (4.82) really is $\frac{1}{2}$.
(*iii*) The kinetic fixed point with an anomalous reaction rate (4.48) corresponds to

$$\begin{aligned} h(x,y) &= 1 + 2xy\left\{1 - \frac{\epsilon + 3\Delta}{3}\left[1 - \gamma - \ln\left(\sqrt{17} - 1\right)x\right]\right\}, \\ \alpha &= 1 + \frac{3\Delta + \epsilon}{3 - \epsilon}\,, \end{aligned} \tag{4.83}$$

with an exact value of the decay exponent.
(*iv*) At the kinetic fixed point with mean-field-like reaction rate (4.51) we obtain

$$h(x,y) = 1 + 2xy\,,$$
$$\alpha = 1\,. \tag{4.84}$$

In the actual asymptotic expression corresponding to (4.79) the argument $y \to \widetilde{C}_n \overline{y}$ is different from that of the Gaussian fixed point.

To complete the picture, we recapitulate – with a little bit more detail – the asymptotic behavior of the number density in the physical space dimension $d = 2$ predicted within the present approach [51] (it turns out that for these conclusions the one-loop calculation is sufficient). On the ray $\varepsilon \le 0$, $\Delta = 0$ logarithmic corrections to the mean-field decay take place. The integral determining the asymptotic behavior of the variable (4.75) yields in

this case

$$\int_\tau^t \frac{\gamma_4 ds}{(2-\gamma_1)s} \underset{t\to\infty}{\sim} -\frac{1}{2}\ln\ln\left(\frac{t}{\tau}\right) + \tilde{c}_4(\tau), \tag{4.85}$$

with corrections vanishing in the limit $t \to \infty$. Therefore, in the vicinity of the fixed point

$$\overline{\lambda}\overline{u}\,\frac{\overline{n}_0}{\mu^d} \sim \lambda u\,\frac{n_0}{\mu^d}\left(\frac{t}{\tau}\right)\ln^{-1/2}\left(\frac{t}{\tau}\right)\widetilde{C}_n \equiv \overline{y}\,\widetilde{C}_n\,. \tag{4.86}$$

The scaling function h is of the simple form

$$h(x,y) = 1 + 2xy$$

and gives rise to asymptotic decay slower than in the mean-field case by a logarithmic factor:

$$n \sim \frac{\ln^{1/2}(t/\tau)}{2v\lambda u\widetilde{C}_n t}.$$

It is worth noting that this logarithmic slowing down is weaker than that brought about the density fluctuations only [57] and this change is produced even by the ubiquitous thermal fluctuations of the fluid, when the reaction is taking place in gaseous or liquid media.

On the open ray $\varepsilon > 0$, $\Delta = 0$ the kinetic fixed point with mean-field-like reaction rate (4.51) is stable and the asymptotic behaviour is given by (4.84) regardless of the value of the falloff exponent of the random forcing in the Navier-Stokes equation. In particular, only the amplitude factor in the asymptotic decay rate in two dimensions is affected by the developed turbulent flow with Kolmogorov scaling, which corresponds to the value $\varepsilon = 2$. This is in accord with the results obtained in the case of quenched solenoidal flow with long-range correlations [63, 68] as well as with the usual picture of having the maximal reaction rate in a well-mixed system.

4.6 CONCLUSION

In conclusion, we have analyzed the effect of density and velocity fluctuations on the reaction kinetics of the single-species decay $A+A \to \emptyset$ universality class in the framework of field-theoretic renormalization group and calculated the scaling function and the decay exponent of the mean particle density for the four asymptotic patterns predicted by the RG.

We have calculated the relevant renormalization constants at two-loop level and found the decay exponent of the mean particle density at this order of the ϵ, Δ expansion for

four IR stable fixed points of the RG, whose regions of stability cover the whole parametric space in the vicinity of the origin in the ϵ, Δ plane. The decay exponent assumes the mean-field value in the basins of attraction of the trivial fixed point (4.43) and of the kinetic fixed point (4.51) with dominant fluctuations of the random force of the Navier-Stokes equation. At the kinetic fixed point with finite rate coefficient (4.48) the decay value of the decay exponent is determined exactly by the fixed-point equations. At the thermal (short-range) fixed point (4.45) the decay exponent possesses a non-trivial ϵ, Δ expansion. We have calculated three first terms of this expansion.

Using a variational approach, we have inferred a renormalized fluctuation-amended rate equation with the account of one-loop corrections. This non-linear integro-differential equation has been solved iteratively in the framework of the ϵ, Δ expansion and the scaling function for the mean particle density has been calculated for the four IR stable regimes. The scaling function assumes the mean-field form (exactly) in the basins of attraction of the trivial fixed point and the kinetic fixed point with dominant fluctuations of the random force. At the kinetic fixed point with finite rate coefficient and at the thermal fixed point the scaling function possesses a non-trivial ϵ, Δ expansion, which we have calculated at the linear order. Fluctuations of the random advection field affect heavily the long-time asymptotic behavior of the system: the kinetic fixed points are brought about by the velocity fluctuations as well as the non-trivial series expansion of the decay exponent at the thermal fixed point (without velocity fluctuations, the decay exponent is fixed to the one-loop value, because there are no high-order corrections to the rate constant in this case). Predictions of the renormalization-group analyses for the reaction $A + A \to \emptyset$ in quenched random fields have been corroborated by numerical simulations [67, 68]. In the case of dynamically generated random drift this is seems to be a much more demanding task, but would surely be highly desirable, since the experimental data for reaction processes is quite scarce.

5 ANNIHILATION PROCESS IN THE PRESENCE OF RANDOM VELOCITY FIELD

5.1 INTRODUCTION

The most realistic description of the velocity field $\mathbf{v}(t,\mathbf{x})$ is based on the use of stochastic Navier-Stokes equation (2.1). However, it is also possible to study a simplified model in which the statistical properties of the advective velocity field are prescribe by hand. One of such models is so-called Kraichnan model introduced in [69]. It assumes Gaussian statistics for the velocity field with self-similar behavior in space and delta correlated in time variable. Despite the its simplicity the model reproduce some of the experimental observations [15] as two-thirds law for second order velocity increments or intermittent behavior for higher order correlation functions. Althought the model can be solved within the zero-mode approach [70, 71, 72, 73], within which exact values for the anomalous dimensions are obtained. So the use of RG method for tackling such problem can be advantegous ,because it allows for comparing results from various independent methods. Another importance of Kraichnan model results from the fact that it is its relatively simple to include and study effects as finite correlation time [21], compressibility[74] or anisotropies [75]. In this section we will concentrate on the effect of Kraichnan model with finite correlation time on the time evolution of the annihilation process $A+A\rightarrow\varnothing$. Physical quantities such as renormalization constants, coordinates of fixed points of renormalization group and corresponding regions of stability are presented to the second (next-to-leading) order approximation.

5.2 FIELD-THEORETIC MODEL

Kraichnan model is based on the assumption that $\mathbf{v}(x)$ is a random Gaussian variable with zero mean and the correlator

$$\langle v_i(t,\mathbf{x})v_j(t',\mathbf{x}')\rangle = \int \frac{d\mathbf{k}d\omega}{(2\pi)^{d+1}} P_{ij}(\mathbf{k})D_v(\omega,\mathbf{k})\exp[-i\omega(t-t')+i\mathbf{k}.(\mathbf{x}-\mathbf{x}')]. \tag{5.1}$$

Here again $P_{ij}(\mathbf{k})=\delta_{ij}-k_ik_j/k^2$ is the transverse projection operator, $k=|\mathbf{k}|$ is the wave number and the kernel function D_v is assumed to have the following form

$$D_v(\omega,\mathbf{k}) = \frac{g_0D_0^3k^{2-2\Delta-2\varepsilon-\eta}}{\omega^2+u_0^2D_0^2(k^{2-\eta})^2}. \tag{5.2}$$

Here g_0 is the coupling constant (small parameter of the ordinary perturbation theory) and the exponents ϵ,Δ and η play the role of small expansion parameters. They could be regarded as an analog of the expansion parameter $\epsilon = 4 - d$ in the language of critical phenomena. However, now exponent ϵ should be understood as deviation from the power law from that of the Kolmogorov scaling [15], whereas Δ is defined as the deviation from the space dimension two via relation $d = 2+2\Delta$. The exponent η is related to the reciprocal of the correlation time at the wave number k. The parameter u_0 may be used for labelling of the fixed points and has the meaning of the ratio of velocity correlation time and the scalar turnover time [76]. It also can be shown [36] that exponent ϵ describes scaling behavior of energy spectrum

$$E(k) \simeq k^{d-1} \int d\omega D_v(\omega, k) \simeq \frac{g_0 D_0^2}{u_0} k^{1-2\epsilon}. \tag{5.3}$$

The Kolmogorov's five-thirds law [15] for spatial velocity statistics is then obtained for the choice $\epsilon = 4/3$ and the choice $\eta = 4/3$ corresponds to the Kolmogorov's frequency.
It is interesting to note that the model (5.1) for the advection field $\mathbf{v}(x)$ contains two cases of special interest:
(a) in the limit $u_0 \to \infty, g_0' \equiv g_0/u_0^2 = const$ we get the 'the rapid-change model'

$$D_v(\omega, \mathbf{k}) \to g_0' D_0 k^{-2-2\Delta-2\epsilon+\eta}, \tag{5.4}$$

which is characterized by the white-in-time nature of the velocity correlator.

(b) limit $u_0 \to 0, g_0'' \equiv g_0/u_0 = const$ corresponds to the case of a frozen velocity field

$$D_v(\omega, \mathbf{k}) \to g_0'' D_0^2 \pi \delta(\omega) k^{2\Delta-2\epsilon}, \tag{5.5}$$

when the velocity field is quenched (time-independent). The presence of delta function $\delta(\omega)$ implies, that correlation function $\langle \mathbf{vv} \rangle$ is independent on the time variable

The averaging procedure with respect to the velocity field $\mathbf{v}(x)$ may be performed with the aid of the following action functional

$$S_2 = -\frac{1}{2} \int dt\, dt'\, d\mathbf{x}\, d\mathbf{x}'\, \mathbf{v}(t, \mathbf{x}) D_v^{-1}(t - t', \mathbf{x} - \mathbf{x}') \mathbf{v}(t', \mathbf{x}'), \tag{5.6}$$

where D_v^{-1} is the inverse correlator (5.1) (in the sense of the Fourier transform). The expectation value of any relevant physical observable may be calculated using the complete

weight functional $\mathscr{W}(\psi^\dagger,\psi,\mathbf{v}) = \mathrm{e}^{S_1+S_2}$, where S_1 and S_2 are the action functionals (4.3) and (5.6). Because now we have to deal only with one velocity field v, the present case is more elementary, in the technical sense than the model given by (4.17). No apperance of response field $\tilde{v}$ means that propagator $\langle\tilde{v}v\rangle$ and interaction vertex $\tilde{v}vv$ are now missing. It directly leads to the great reduce in the number of possible two-loops graphs for the 1PI function $\langle\psi^\dagger\psi\rangle$. The propagators for the model given as sum of the actions (4.3) and (5.6) are given as

$$\begin{aligned}\langle vv\rangle_0 &= \frac{g_0 D_0^3 k^{2-2\delta-2\epsilon-\eta}}{\omega^2+u_0^2 D_0^2 (k^{2-\eta})^2} P_{ij}(\mathbf{k}),\\ \langle\psi\psi^\dagger\rangle_0 &= \langle\psi^\dagger\psi\rangle_0^* = \frac{1}{-i\omega_k + D_0 k^2}, \quad \langle\psi\psi\rangle_0 = \langle\psi^\dagger\psi^\dagger\rangle_0 = 0\end{aligned} \tag{5.7}$$

in the frequency-momentum $(\omega,\mathbf{k})$ representation.

5.3 POWER COUNTING AND UV RENORMALIZATION

In order to apply the minimal subtraction scheme for the evaluation of renormalization constants, an analysis of possible superficial divergences has to be performed. We proceed similartly as was done in Sec. 4.3. The canonical analysis is performed in

Q	ψ	$\psi^\dagger$	v	D_0	u_0	λ	g_0
d_Q^k	d	0	-1	-2	η	-2Δ	$2\epsilon+\eta$
d_Q^ω	0	0	1	1	0	0	0
d_Q	d	0	1	0	η	-2Δ	$2\epsilon+\eta$

Table 4: Canonical dimensions for the parameters

straightforward manner and the obtained values for fields and parameters in d-dimensional space are given in Tab. 4. The total canonical dimension $d_Q = 2d_Q^\omega + d_Q^k$ is determined from the condition that the parabolic differential operator of the diffusion scales uniformly under the transformation $k \to \mu k, \omega \to \mu^2\omega$.

The model is logarithmic for $\epsilon = \Delta = 0$, the UV divergences have the form of poles in various linear combinations of ϵ and Δ. The total canonical dimension of an arbitrary 1PI function $\Gamma = \langle\Phi\ldots\Phi\rangle_{1-\mathrm{ir}}$ is given by relation

$$d_\Gamma = d + 2 - N_\Phi d_\Phi, \tag{5.8}$$

where $N_\Phi = \{N_{\psi^\dagger}, N_\psi, N_v\}$ are the numbers of corresponding external fields. Superficial UV divergences may be present only in those Γ functions for which d_Γ is a non-negative integer. Although the canonical dimension of the field $\psi^\dagger$ is zero, there is no proliferation of superficial divergent graphs with arbitrary number of external $\psi^\dagger$ legs. This is due to the fact that $n_{\psi^\dagger} \leq n_\psi$, which is easily seen by a straightforward analysis of the graphs [24]. As has already been shown [76] that the divergences in (1PI) Green functions containing at least one velocity field $\mathbf{v}$ could be removed by the only counterterm of the form $\psi^\dagger \partial^2 \psi$, which leads to the following renormalization of parameters g_0, D_0 and u_0

$$D_0 = D Z_D, \qquad g_0 = g \mu^{2\epsilon+\eta} Z_g, \qquad u_0 = u \mu^\eta Z_u, \tag{5.9}$$

where μ is the reference mass scale in the minimal subtraction (MS) scheme. The renormalization constants Z_g, Z_u and Z_D satisfy the relations

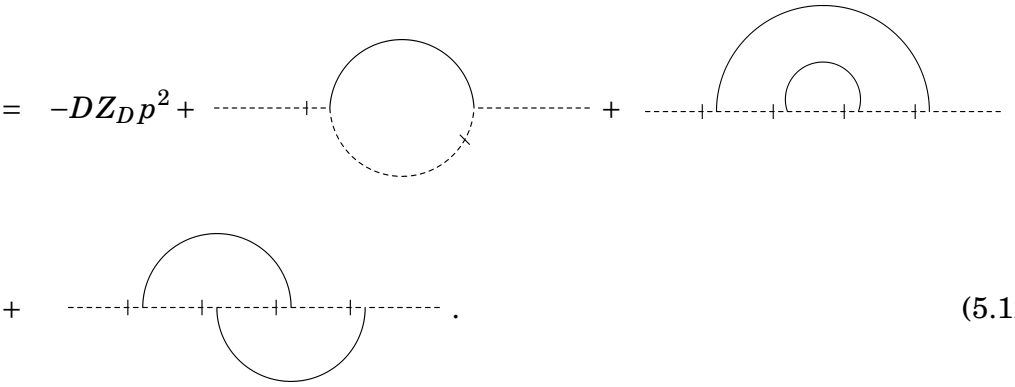
$$Z_g = Z_D^{-3}, \qquad Z_u = Z_D^{-1}. \tag{5.10}$$

It should be noted that the graphs corresponding to $\Gamma_{\psi^\dagger\psi\psi}$ and $\Gamma_{\psi^\dagger\psi^\dagger\psi\psi}$ differ only by one external vertex and thus give rise to equal renormalization of the rate constant $\lambda_0 D_0$. Singularities presented in these (1PI) Green functions may be eliminated by the following renormalization of λ_0

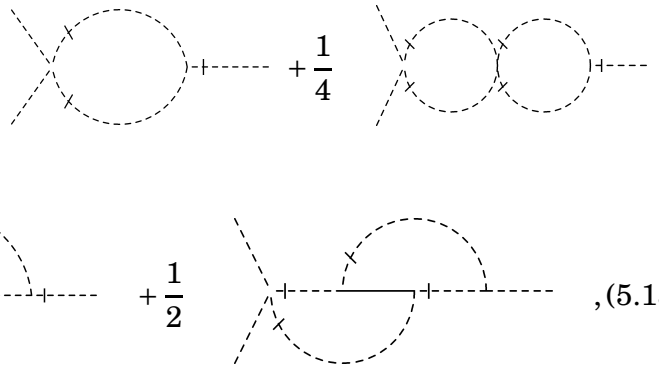
$$\lambda_0 = \lambda \mu^{-2\Delta} Z_D^{-1} Z_\lambda. \tag{5.11}$$

The explicit form of the renormalization constants Z_D and Z_λ at the two-loop order can be found from the condition of ultraviolet finitness for the Green functions $\Gamma_{\psi^\dagger\psi}$ and $\Gamma_{\psi^\dagger\psi^2}$ (or $\Gamma_{\psi^{\dagger 2}\psi^2}$ respectively). Because we know that the UV divergent part of $\Gamma_{\psi^\dagger\psi}$ does not depend on the time variable, we choose the simplest choice $\omega = 0$ and the Dyson equation

for $\Gamma_{\psi^\dagger\psi}$ can be schematically written as

$\Gamma_{\psi^\dagger\psi}|_{\omega=0} = -DZ_D p^2 +$  (5.12)

Finally the perturbation expansion for the 1PI function $\Gamma_{\psi^\dagger\psi^2}$ can be written as

$\Gamma_{\psi^\dagger\psi^2}|_{\omega=0,p^2=0} = -4D\lambda Z_\lambda \mu^{-2\Delta} + \frac{1}{2}$  (5.13)

where the prefactor represents corresponding symmetry coefficient of the following Feynman graph. The explicit expressions for the singular parts of the Feynman diagrams can be found in Appendix (B). We again stress that convergent part are neglected in the MS scheme [53]. Two loop calculation leads to the following results for the renormalization constants Z_D and Z_λ

$$
\begin{aligned}
Z_D &= 1 - \frac{\overline{g}}{2\epsilon} + A(u)\frac{\overline{g}^2}{2\epsilon} - \frac{1}{2(1+u)}\frac{\overline{g}^2}{(2\epsilon)^2} \\
A(u) &= \frac{1}{8(1+u)}\left[4\xi + 3 + u + u(u+2)(u+3)\ln\frac{u^2+2u}{(1+u)^2}\right]
\end{aligned}
\tag{5.14}
$$

$$
\begin{aligned}
Z_\lambda &= 1-\frac{\overline{\lambda}}{2\Delta}+\frac{\overline{\lambda}^2}{(2\Delta)^2}+B(u)\frac{\overline{g\lambda}}{2(\epsilon-\Delta)}-\frac{\overline{g\lambda}}{4\Delta(\epsilon-\Delta)} \\
B(u) &= -\frac{1}{2}\Big[\xi+1+\ln\frac{1+u}{2}+(2u+1)\ln\frac{2u+1}{2u+2}\Big]+\frac{2(1+u)}{\pi}\int_{-1}^{1}dz\,F(z) \\
F(z) &= \frac{(1-z^2)^{1/2}}{(1-u)^2+4uz^2}\Big\{\frac{u-1}{2}\ln\frac{1+u}{2}-\frac{2(1+u)z}{\sqrt{1-z^2}}\Big[\frac{\pi}{2}-\arctan\sqrt{\frac{1+z}{1-z}}\Big]+ \\
&\frac{(u+3)z}{\sqrt{2(1+u)-z^2}}\Big[\pi-\arctan\frac{z+u+1}{\sqrt{2(1+u)-z^2}}-\arctan\frac{(2+z)}{\sqrt{2(1+u)-z^2}}\Big]\Big\}, \qquad (5.15)
\end{aligned}
$$

where for brevity we have used the following normalization of charges

$$
\overline{g}=\frac{g}{4u(1+u)}\frac{S_d}{(2\pi)^d}, \quad \overline{\lambda}=\lambda\frac{S_d}{(2\pi)^d}, \quad S_d=\frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}.
$$

Let just note that the obtained results (5.14) and (5.15) are presented directly for the space dimension $d=2$.

5.4 FIXED POINTS

The coefficient functions of the RG operator

$$
D_{\rm RG}=\mu\frac{\partial}{\partial\mu}\Big|_0=\mu\frac{\partial}{\partial\mu}+\sum_{g_i}\beta_i\frac{\partial}{\partial g_i}-\gamma_D D\frac{\partial}{\partial D}, \qquad (5.16)
$$

where the bare parameters are denoted with the subscript "0", are defined as

$$
\gamma_D=\mu\frac{\partial\ln Z_D}{\partial\mu}\Big|_0, \quad \beta_i=\mu\frac{\partial g_i}{\partial\mu}\Big|_0, \qquad (5.17)
$$

with the charges $g_i=\{g,u,\lambda\}$. From this definition and from relations (5.9),(5.10) and (5.11) it follows that

$$
\beta_g=g(-2\epsilon-\eta+3\gamma_D), \quad \beta_u=u[-\eta+\gamma_D], \quad \beta_\lambda=\lambda(2\Delta-\gamma_\lambda+\gamma_D) \qquad (5.18)
$$

The scaling regimes are associated with the fixed points of the corresponding RG functions. The fixed points are defined as such points g^*,u^*,λ^* for which the β functions vanish

$$
\beta_g(g^*,u^*,\lambda^*)=\beta_u(g^*,u^*,\lambda^*)=\beta_\lambda(g^*,u^*,\lambda^*)=0. \qquad (5.19)
$$

The type of the fixed point is determined by the eigenvalues of the matrix

$$\Omega \equiv \{\Omega_{ik}\} = \frac{\partial \beta_i}{\partial g_k}, \tag{5.20}$$

where β_i is the full set of β functions (5.18) and g_k is the full set of charges $\{g, u, \lambda\}$. The IR asymptotic behavior is governed by the IR stable fixed points, for which all eigenvalues of Ω matrix are positive.

It is easy to see that the functions β_g and β_u satisfy relation $\beta_g/g - 3\beta_u/u = 2(\eta - \epsilon)$. This means that they cannot be equal zero simultaneously for the finite values of the charges g and u. The only exception is the instance $\epsilon = \eta$, which should be studied separately. For general case $\epsilon \neq \eta$ we have to set either $u = 0$ or $u = \infty$ and rescale g in such a way, that γ_D remains finite [21].

In what follows we present the results for fixed points, anomalous dimensions and eigenvalues of the Ω matrix to the second order of perturbation theory. However, we would like to stress, that the form of β functions (5.18) allows to calculate the anomalous dimensions γ_D and γ_λ exactly (without any second-order correction).

In [76] independence of the renormalization constant Z_D on the exponents η at the two-loop approximation has been conjectured. It implies that we may use the choice $\eta = 0$, which we have applied in our calculations of the renormalization constants Z_D and Z_λ.

Let us consider the "rapid-change mode" ($u \to \infty$). It is convenient to introduce new variables $w = 1/u, g' = g/u^2$ and the corresponding β functions obtain the form

$$\beta_{g'} = g'[\eta - 2\epsilon + \gamma_D], \quad \beta_w = w[\eta - \gamma_D], \quad \beta_\lambda = \lambda(2\Delta - \gamma_\lambda + \gamma_D). \tag{5.21}$$

The "rapid-change model" corresponds to the fixed point with $w^* = 0$. In this case four stable IR fixed points are realized:

$$\begin{aligned} \textbf{FP 1A:} \quad & w^* = 0, \quad g'^* = 0, \quad \lambda^* = 0 \\ & \gamma_D = 0, \quad \gamma_\lambda = 0 \\ & \Omega_1 = \eta - 2\epsilon, \quad \Omega_2 = \eta, \quad \Omega_3 = 2\Delta \end{aligned} \tag{5.22}$$

$$\begin{aligned} \textbf{FP 1B:} \quad & w^* = 0, \quad g'^* = 0, \quad \lambda^* = -4\pi\Delta \\ & \gamma_D = 0, \quad \gamma_\lambda = 2\Delta \\ & \Omega_1 = \eta - 2\epsilon, \quad \Omega_2 = \eta, \quad \Omega_3 = -2\Delta \end{aligned} \tag{5.23}$$

(5.24)

$$\textbf{FP 2A:} \quad w^* = 0, \quad g'^* = 8\pi(2\epsilon - \eta), \quad \lambda^* = 0$$
$$\gamma_D = 2\epsilon - \eta, \quad \gamma_\lambda = 0 \tag{5.25}$$
$$\Omega_1 = 2\epsilon - \eta, \quad \Omega_2 = 2\eta - 2\epsilon, \quad \Omega_3 = 2\Delta + 2\epsilon - \eta$$
$$\textbf{FP 2B:} \quad w^* = 0, \quad g'^* = 8\pi(2\epsilon - \eta),$$
$$\lambda^* = -2\pi(2\Delta + 2\epsilon - \eta) - \pi\frac{2\Delta + 2\epsilon - \eta}{\epsilon}(2\epsilon - \eta)(\Delta + \epsilon)$$
$$\gamma_D = 2\epsilon - \eta, \quad \gamma_\lambda = 2\Delta + 2\epsilon - \eta \tag{5.26}$$
$$\Omega_1 = 2\epsilon - \eta, \quad \Omega_2 = 2\eta - 2\epsilon, \quad \Omega_3 = -2\Delta - 2\epsilon + \eta.$$

For the analysis of the regime $u \to 0$ (quenched velocity field) we introduce the new variable $g'' \equiv g/u$. Hence the corresponding β functions have the form

$$\beta_{g''} = g''[-2\epsilon + 2\gamma_D], \quad \beta_u = u[-\eta + \gamma_D], \quad \beta_\lambda = \lambda(2\Delta - \gamma_\lambda + \gamma_D) \tag{5.27}$$

Also in this case there are four possible IR stable fixed points:

$$\textbf{FP 3A:} \quad u^* = 0, \quad g''^* = 0, \quad \lambda^* = 0$$
$$\gamma_D = 0, \quad \gamma_\lambda = 0 \tag{5.28}$$
$$\Omega_1 = -2\epsilon, \quad \Omega_2 = -\eta, \quad \Omega_3 = 2\Delta$$
$$\textbf{FP 3B:} \quad u^* = 0, \quad g''^* = 0, \quad \lambda^* = -4\pi\Delta$$
$$\gamma_D = 0, \quad \gamma_\lambda = 2\Delta \tag{5.29}$$
$$\Omega_1 = -2\epsilon, \quad \Omega_2 = -\eta, \quad \Omega_3 = -2\Delta$$
$$\textbf{FP 4A:} \quad u^* = 0, \quad g''^* = 8\pi\epsilon + 2\pi(4\Delta + 3\epsilon)\epsilon, \quad \lambda^* = 0$$
$$\gamma_D = \epsilon, \quad \gamma_\lambda = 0 \tag{5.30}$$
$$\Omega_1 = 2\epsilon - \frac{4\Delta + 3\epsilon}{2}\epsilon, \quad \Omega_2 = -\eta + \epsilon, \quad \Omega_3 = 2\Delta + \epsilon$$
$$\textbf{FP 4B:} \quad u^* = 0, \quad g''^* = 8\pi\epsilon + 2\pi(4\Delta + 3\epsilon)\epsilon,$$
$$\lambda^* = -2\pi(2\Delta + \epsilon) + \pi(2\Delta + \epsilon)(\mathrm{Q}\epsilon - \Delta)$$
$$\gamma_D = \epsilon, \quad \gamma_\lambda = 2\Delta + \epsilon \tag{5.31}$$
$$\Omega_1 = 2\epsilon - \frac{4\Delta + 3\epsilon}{2}\epsilon, \quad \Omega_2 = -\eta + \epsilon, \quad \Omega_3 = -2\Delta - \epsilon,$$

where Q = 1.159 is a numerical constant resulting from a simple but cumbersome two-loop integral.

In the special case $\epsilon = \eta$ the functions β_g and β_u become proportional and this leads to the degeneration of fixed point. Instead of just plain fixed point, we have a line of fixed

points in the $g-u$ plane.

$$\begin{aligned}
&\textbf{FP 5A:} \quad \lambda^* = 0 \\
&\gamma_D = \epsilon, \quad \gamma_\lambda = 0 \\
&\Omega_1 = G_1(u^*)\epsilon + G_2(u^*)\epsilon, \quad \Omega_2 = 0, \quad \Omega_3 = 2\Delta + \epsilon
\end{aligned} \tag{5.32}$$

$$\begin{aligned}
&\textbf{FP 5B:} \quad \lambda^* \neq 0 \\
&\gamma_D = \epsilon, \quad \gamma_\lambda = 2\Delta + \epsilon \\
&\Omega_1 = G_1(u^*)\epsilon + G_2(u^*)\epsilon, \quad \Omega_2 = 0, \quad \Omega_3 = -2\Delta - \epsilon
\end{aligned} \tag{5.33}$$

where the functions G_1 and G_2 are given as

$$\begin{aligned}
G_1(u^*) &= \frac{u^*+2}{u^*+1} \\
G_2(u^*) &= -\frac{1}{4(1+u^*)}\left[u(3u^3+15u^2+26u+18)\ln\frac{u^2+2u}{(1+u)^2} + 2(4\xi+3) + 3u^2 + 9u\right].
\end{aligned} \tag{5.34}$$

The zero eigenvalue Ω_2 is connected with the existence of a marginal direction in the $g-u$ plane (along the line of the fixed points).

The "real problem" corresponds to the value $\epsilon = 4/3$, which leads to the famous Kolmogorov "five-thirds law" [15] for the spatial velocity statistics. By direct numerical calculation it may be easily shown that in this case ($\epsilon = 4/3$) the eigenvalue Ω_1 is positive. Of course this holds just for the physically possible values of parameters u and ξ. One has to consider $u > 0$ and $\xi < 0$, which corresponds to the space dimensions lesser than two. The conclusion is that all fixed points on the line are IR stable.

The phase diagrams in (ϵ, Δ)-plane ($\eta = 0$) and in ϵ, η-plane ($\Delta = 0$) are depicted in Figures 13 and 14 in one-loop approximation. The broad lines represents boundaries between given regimes.

FP	1A	1B	2A	2B	3A	3B	4A	4B	5A	5B
α	1	$1+\Delta$	1	$\frac{2+2\Delta}{2-2\epsilon+\eta}$	1	$1+\Delta$	1	$\frac{2+2\Delta}{2-\epsilon}$	1	$\frac{2+2\Delta}{2-\epsilon}$

Table 5: Decay exponent

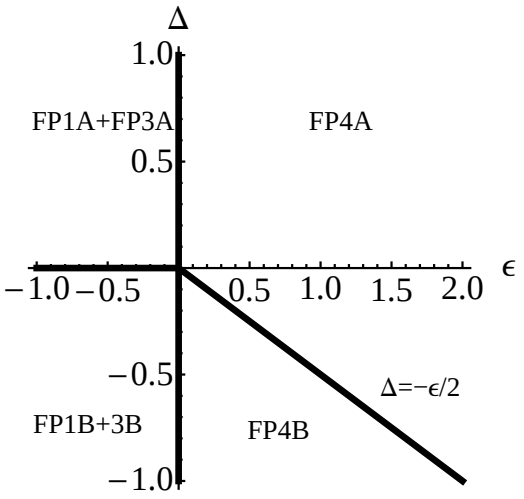


Figure 11: Phase diagram in the (ϵ, Δ) plane for $\eta = 0$

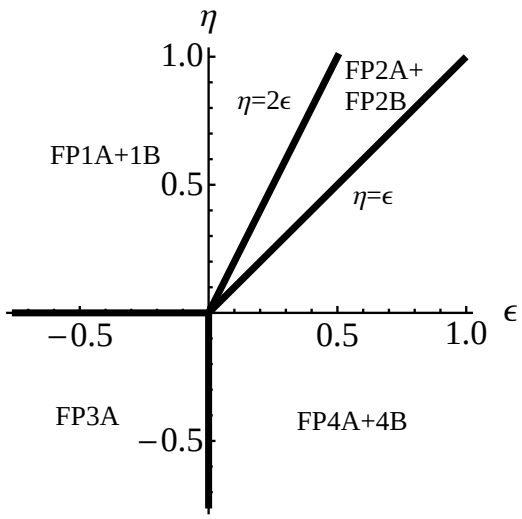


Figure 12: Phase diagram in the (ϵ, η) plane for $\Delta = 0$ ($d = 2$).

5.5 LONG-TIME ASYMPTOTICS OF CONCENTRATION

Applying the result (4.78) for the present model it is possible to find decaying exponent α for all of fixed points **1A** – **5B**. Their actual values are listed in Tab. 5. The fixed points **1A** and **3A** corresponds to the mean-field regimes, that predicts correct scaling behavior above critical space dimension $\Delta > 0$. For fixed points **1B** and **3B** only density fluctuations are important (advection can be effectively neglected) and result [24] is recovered. In the regimes governed by **FP2B** and **FP4B**, when both effects (advection plus reaction itself) are important, it is easy to see that that the value of β is larger than diffusion limited value $1 + \Delta$ but smaller than mean-field prediction 1. For the turbulent case $\eta = \epsilon = 4/3$ the value $\beta = 3(1 + \Delta)$ (for **5B**) follows. That could also be seen from the scaling analysis mentioned in the Sec.3.1 employing know scaling behavior for mean radius of gyration [36].

6 INFLUENCE OF COMPRESSIBILITY ON THE ANOMALOUS KINETICS OF THE ANNIHILATION PROCESS

6.1 INTRODUCTION

Kraichnan model as a model for advection of passive scalar can be introduced by the statistical properties (5.1) and (5.2). It is relatively simple model for fluctuations of advecting field $\mathbf{v}_0$ (the subsricpt 0 has been added because it turns out that adding of compressibility leads to the renormalization of the velocity field) in the sense, that it doesn't take into account properties of real fluids. Often large differences are observed in relation to the experiment and their origin could be caused by large-scale anisotropies, effect of compressibility or parity violation (helicity).

Study [77] shows that there are important differences between advection of scalar quantity (temperature, concentration, tracer) by compressible versus incompressible fluid. It was shown that compressibility could lead to the slowing of transport process for scalar admixture and also to the enhancement of intermittency. These effects might be understood as a result of inhibition of separation between particle trajectories and therefore we expect that reacting particles would spent effectively more time in the mutual vicinity than in the incompressible case. Hence in this case we expect the faster decay rate than for the incompressible case.

In this section we give a brief account on the effect of compressibility on the single species annihilation process $2A \to \varnothing$. Again by using perturbative renormalization group approach the model is renormalized near its critical dimension $d_2 = 2$. In the one-loop order all relevant physical quantities are calculated and contrary to the previous sections we found out that already in the one-loop approximation fluctuations of the velocity field affect renormalization of the rate constant and thus contribute to the value of decaying exponent (4.78).

After brief description of the model in Sec. 6.2, we present results of RG calculations to the one-loop order in Sec. 6.3. In Sec. 6.4 possible large-scale regimes are listed and their physical interpretation is concluded in Sec. 6.5.

6.2 FIELD-THEORETIC MODEL

Similarly to the previous chapter 5 we consider gaussian ensemble for the generation of velocity fluctuations, but now with inclusion of the term responsible for the violation of incompressibility condition $\nabla.\mathbf{v} = 0$. It can be explained by the presence of longitudinal

(in the Fourier representation $\mathbf{v}_0 \propto \mathbf{k}$) components of the velocity field $\mathbf{v}_0(t,\mathbf{x})$. The correlator for velocity field $\mathbf{v}$ is considered to be a Gaussian random variable with zero mean and with correlator [74] given as

$$\langle v_{0i}(x)v_{0j}(x')\rangle = \int \frac{d\mathbf{k}d\omega}{(2\pi)^{d+1}}[P_{ij}(\mathbf{k}) + \alpha Q_{ij}(\mathbf{k})]D_v(\omega,\mathbf{k})\exp[-i\omega(t-t') + i\mathbf{k}.(\mathbf{x}-\mathbf{x}')], \quad (6.1)$$

where the longitudinal projector Q has been introduced

$$Q_{ij}(\mathbf{k}) = \frac{k_i k_j}{k^2}. \quad (6.2)$$

The positive parameter (necessary condition to have positive defined correlator $\langle vv\rangle$) α represents degree of compressibility. The incompressible case is obtained by the setting $\alpha = 0$ and in this way numerical results can be checked for consistence.
The inclusion of projector Q doesn't affect the theoretical construction developed for the Kraichnan model introduced previously (see chapter 5). Therefore we just mention main steps of theoretical description and deviations from it caused by compressibility violation. Action functional responsible for the inclusion of velocity fluctuation is of usual Gaussian form given by (5.6). Fourier transformation of velocity propagator $\langle vv\rangle$ is easily obtained

$$\langle v_{0i}v_{0j}\rangle_0 = \frac{g_0 D_0^3 k^{2-2\Delta-2\epsilon-2\eta}}{\omega^2 + (u_0 D_0 k^{2-\eta})^2}[P_{ij}(\mathbf{k}) + \alpha Q_{ij}(\mathbf{k})]. \quad (6.3)$$

Obviously the parameter α is dimensionless and all canocical dimensions of fields and parameters are given in Tab. 4. The only difference with the incompressible case is that now the velocity field has to be renormalized [74] according to prescription

$$\mathbf{v}_0 = \mathbf{v}Z_v. \quad (6.4)$$

The bare action for the annihilation process in the presence of advection is given by the (4.3) and the action for the velocity field is Gaussian (5.6). The renormalization of the complete model is then achieved by the following relations

$$D_0 = DZ_D, \quad g_0 = g\mu^{2\epsilon+\eta}Z_g, \quad u_0 = u\mu^{\eta}Z_u, \quad \lambda_0 = \lambda\mu^{-2\Delta}Z_D^{-1}Z_\lambda, \quad \mathbf{v}_0 = \mathbf{v}Z_v \quad (6.5)$$

with the additional constraints between them

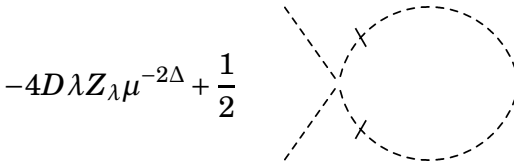
$$Z_g Z_D^3 = 1, \quad Z_u Z_D = 1, \quad Z_g Z_D^3 = Z_{\mathbf{v}}^2. \quad (6.6)$$

They are implied by the general statement of the renormalization theory [1], that states non-renormalization of the non-local terms in the action. The velocity correlator is very non-local because of nontrivial correlations both in momentum and frequency scales. The total renormalized action can be written as

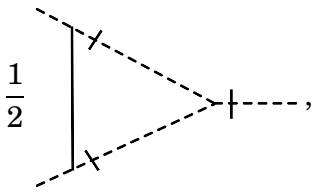
$$S_R(\psi^\dagger,\psi,\mathbf{v}) = \psi^\dagger[-\partial_t\psi + DZ_D\nabla^2\psi - Z_v(\mathbf{v}.\boldsymbol{\nabla})\psi] - \frac{vD_v^{-1}v}{2} - Z_\lambda D\lambda[2\psi^\dagger + \psi^{\dagger 2}]\psi^2 \,. \quad (6.7)$$

The perturbative calculation of renormalization constants in dimensional regularization with the use minimal subtraction scheme is straighforward. We restrict oureselves to the first order in perturbation theory and it is physically interesting fact that this approximation already contains first nontrivial effect of compressibility itself. It can be seen from the perturbation serie for the 1PI function $\Gamma_{\psi^\dagger\psi^2}$ that can be written in the form

$$\Gamma_{\psi^\dagger\psi^2}|_{\omega=0,p^2=0} = -4D\lambda Z_\lambda \mu^{-2\Delta} + \frac{1}{2} \quad + \quad (6.8)$$



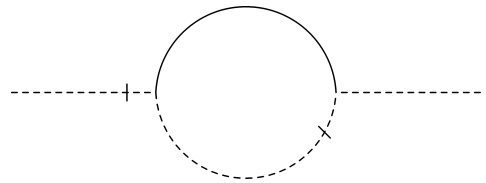
$$\frac{1}{2} \quad , \quad (6.9)$$



where the symmetry coefficients are explicitly emphasized. The first graph physically represents process of density fluctuations that account for annihilating of two particles. It is easy to see that transverse character (equivalent statement to the incompressibility condition of velocity field $\mathbf{v}$) of propagator $\langle vv\rangle$ leads to UV divergent contribution of the second graph in (6.9), whereas for the models with incompressible velocity field $\mathbf{v}$ it leads to UV convergent contribution. Physically this graph can be explained in a terms of attracting process that brings together particles into a sink of compression, which effectively increases the reaction rate.

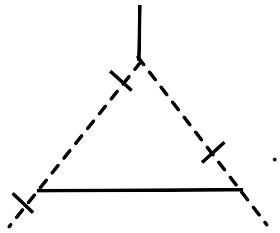
The Dyson equation [1] for propagator $\langle\psi^\dagger\psi\rangle$ can be written as

$$\Gamma_{\psi^\dagger\psi}|_{\omega=0} = -DZ_D p^2 + \quad (6.10)$$



from which constant Z_D is extracted, whereas renormalization constant Z_v for the velocity field is calculated from the triple vertex function

$$\Gamma_{\psi^\dagger\psi v}|_{\omega=0} = ipZ_v +$$



. (6.11)

Here p is the outgoing momentum of the field $\psi^\dagger$. The explicit results for the Feynman diagrams can be found in Appendix C. One loop calculation yields the following results for renormalization constants

$$Z_D = 1 - \frac{g}{16\pi u(1+u)\epsilon}\left[1+\alpha-\frac{2\alpha}{1+u}\right], \quad (6.12)$$

$$Z_v = 1 + \frac{\alpha g}{16\pi u(1+u)^2\epsilon}, \quad (6.13)$$

$$Z_\lambda = 1 - \frac{\lambda}{4\pi\Delta} - \frac{\alpha g}{16\pi u(1+u)\epsilon} \quad (6.14)$$

and from relations (6.6) the form of Z_u and Z_g also follows

$$Z_u = 1 + \frac{g}{16\pi u(1+u)\epsilon}\left[1+\alpha-\frac{2\alpha}{1+u}\right], \quad (6.15)$$

$$Z_g = 1 + \frac{3g}{16\pi u(1+u)\epsilon}\left[1+\alpha-\frac{2\alpha}{1+u}\right]. \quad (6.16)$$

Setting the value $\alpha = 0$ results (5.14) and (5.15) (to the one-loop precision) are recovered together with $Z_v = 1$, that corresponds to the non-renormalizibity of the velocity field. Therefore the incompressible limit is recovered.

6.3 RG FUNCTIONS

From relations (6.5) and (6.6) immediately follows beta functions (also known as flow functions for charges [10])

$$\beta_g = g[-2\epsilon-\eta+3\gamma_D-2\gamma_{\mathbf{v}}], \quad \beta_u = u[-\eta+\gamma_D], \quad \beta_\lambda = \lambda[2\Delta-\gamma_\lambda+\gamma_D], \quad (6.17)$$

and anomalous dimensions γ_F are obtained according the relation

$$\gamma_F = \mu\partial_\mu \ln Z_F = (\beta_g\partial_g + \beta_u\partial_u + \beta_\lambda\partial_\lambda)\ln Z_F \quad (6.18)$$

In the one-loop approximation only first order terms of Taylor expansion of logarithmic has to be retained and therefore relation (6.18) can be simplified

$$\gamma_F = [-2\epsilon g\partial_g + 2\Delta\lambda\partial_\lambda]\ln Z_F, \tag{6.19}$$

which is very suitable form for practical calculations. In the actual calculations of the Feynman graphs one can set $\eta = 0$. It was conjectured in the works [21, 74, 76] that to two-loop order actual value of exponent η has no influence on the renormalization constants. So in the calculation of the UV divergent parts of Feynman diagrams one can simply set $\eta = 0$.

Finally following expressions for the anomalous dimensions are obtained

$$\gamma_D = \frac{g}{8\pi u(1+u)}\Big[1+\alpha-\frac{2\alpha}{u+1}\Big], \quad \gamma_{\mathbf{v}} = -\frac{\alpha g}{8\pi u(1+u)^2}, \quad \gamma_\lambda = \frac{\alpha g}{8\pi u(1+u)} - \frac{\lambda}{2\pi} \tag{6.20}$$

We see that now the velocity field attains non-zero anomalous dimension, that is proportional to the compressiblity parameter α.

6.4 IR STABLE REGIMES

Large scale behavior is governed by the infrared fixed points, that are determined as zero points of beta functions (6.17). The region of stability for the given fixed point is obtained from its matrix of first derivatives (5.20). Also by direct generalizing the result (4.78) for the present model the decaying exponent β : $n(t) \overset{t\to\infty}{\sim} t^{-\beta}$ can be calculated to the (lowest) tree level. Note that here we have to deal with simpler situation because we have only three charges $\{g,u,\lambda\}$ instead of four $\{g_1,g_2,u,\lambda\}$. Detailed analysis of fixed point structure reveals that studied system can exhibit one of ten possible regimes, that are listed below.

First let us consider the "rapid-change mode" ($u \to \infty$) . Using convienent variables $w = 1/u, g' = g/u^2$ and the corresponding β functions obtain the following form

$$\beta_{g'} = g'[-2\epsilon+\eta+\gamma_D-2\gamma_v], \quad \beta_w = w[\eta-\gamma_D], \quad \beta_\lambda = \lambda[2\Delta-\gamma_\lambda+\gamma_D], \tag{6.21}$$

where anomalous dimensions

$$\gamma_D = \frac{g'}{8\pi(1+w)}\Big[1+\alpha-\frac{2\alpha w}{1+w}\Big], \quad \gamma_v = -\frac{\alpha g'}{8\pi(1+w)^2}, \quad \gamma_\lambda = \frac{\alpha g'}{8\pi(1+w)} - \frac{\lambda}{2\pi}. \tag{6.22}$$

The "rapid-change model" corresponds to the fixed point with the value $w^* = 0$. In this

case four stable IR fixed points can be realized:

$$\begin{aligned}
&\textbf{FP 1:} \quad w^* = 0, \quad g'^* = 0, \quad \lambda^* = 0 \\
&\gamma_D = 0, \quad \gamma_v = 0, \quad \gamma_\lambda = 0 \qquad (6.23)\\
&\Omega_1 = \eta - 2\epsilon, \quad \Omega_2 = \eta, \quad \Omega_3 = 2\Delta \\
&\beta = 1, \\
&\textbf{FP 2:} \quad w^* = 0, \quad g'^* = 0, \quad \lambda^* = -4\pi\Delta \\
&\gamma_D = 0, \quad \gamma_v = 0, \quad \gamma_\lambda = 2\Delta \qquad (6.24)\\
&\Omega_1 = \eta - 2\epsilon, \quad \Omega_2 = \eta, \quad \Omega_3 = -2\Delta \\
&\beta = 1 + \Delta, \\
&\textbf{FP 3:} \quad w^* = 0, \quad g'^* = \frac{8\pi(2\epsilon - \eta)}{1+\alpha}, \quad \lambda^* = 0 \\
&\gamma_D = 2\epsilon - \eta, \quad \gamma_v = 0, \quad \gamma_\lambda = 0 \qquad (6.25)\\
&\Omega_1 = 2\epsilon - \eta, \quad \Omega_2 = 2\eta - 2\epsilon, \quad \Omega_3 = 2\Delta + \frac{2\epsilon - \eta}{1+\alpha}, \\
&\beta = \frac{2\alpha + 2 - 2\epsilon + \eta}{(1+\alpha)(2 - 2\epsilon + \eta)} \\
&\textbf{FP 4:} \quad w^* = 0, \quad g'^* = \frac{8\pi(2\epsilon - \eta)}{1+\alpha}, \quad \lambda^* = 2\pi\left(2\Delta + \frac{-2\epsilon + \eta}{1+\alpha}\right) \\
&\gamma_D = 2\epsilon - \eta, \quad \gamma_v = 0, \quad \gamma_\lambda = 2\Delta + 2\epsilon - \eta \qquad (6.26)\\
&\Omega_1 = 2\epsilon - \eta, \quad \Omega_2 = 2\eta - 2\epsilon, \quad \Omega_3 = -2\Delta - \frac{2\epsilon - \eta}{1+\alpha}, \\
&\beta = \frac{2 + 2\Delta}{2 - 2\epsilon + \eta}. \qquad (6.27)
\end{aligned}$$

For the analysis of the regime $u \to 0$ (quenched velocity field) we introduce the new variable $g'' \equiv g/u$. Hence the corresponding β functions have the form

$$\beta_{g''} = g''[-2\epsilon + 2\gamma_D - 2\gamma_v], \quad \beta_u = u[-\eta + \gamma_D], \quad \beta_\lambda = \lambda[2\Delta - \gamma_\lambda + \gamma_D]. \qquad (6.28)$$

and anomalous dimensions are given as

$$\gamma_D = \frac{g''}{8\pi(1+u)}\left[1 + \alpha - \frac{2\alpha}{1+u}\right], \quad \gamma_v = -\frac{\alpha g''}{8\pi(1+u)^2}, \quad \gamma_\lambda = \frac{\alpha g''}{8\pi(1+u)} - \frac{\lambda}{2\pi}. \qquad (6.29)$$

The quenched regime corresponds to the $u^* = 0$ and also in this case there are four possi-

ble IR stable fixed points:

FP 5: $u^* = 0, \quad g''^* = 0, \quad \lambda^* = 0$

$$\gamma_D = 0, \quad \gamma_v = 0, \quad \gamma_\lambda = 0 \tag{6.30}$$

$$\Omega_1 = -2\epsilon, \quad \Omega_2 = -\eta, \quad \Omega_3 = 2\Delta \tag{6.31}$$

$$\beta = 1,$$

FP 6: $u^* = 0, \quad g''^* = 0, \quad \lambda^* = -4\pi\Delta$

$$\gamma_D = 0, \quad \gamma_v = 0, \quad \gamma_\lambda = 2\Delta \tag{6.32}$$

$$\Omega_1 = -2\epsilon, \quad \Omega_2 = -\eta, \quad \Omega_3 = -2\Delta,$$

$$\beta = 1 + \Delta,$$

FP 7: $u^* = 0, \quad g''^* = 8\pi\epsilon, \quad \lambda^* = 0$

$$\gamma_D = \epsilon(1-\alpha), \quad \gamma_v = -\alpha\epsilon, \quad \gamma_\lambda = \alpha\epsilon \tag{6.33}$$

$$\Omega_1 = 2\epsilon, \quad \Omega_2 = -\eta + \epsilon(1-\alpha), \quad \Omega_3 = 2\Delta + \epsilon(1-2\alpha),$$

$$\beta = \frac{2-(1-2\alpha)\epsilon}{2-(1-\alpha)\epsilon},$$

FP 8: $u^* = 0, \quad g''^* = 8\pi\epsilon,$

$$\lambda^* = 2\pi[-2\Delta + \epsilon(2\alpha - 1)]$$

$$\gamma_D = (1-\alpha)\epsilon, \quad \gamma_v = -\alpha\epsilon, \quad \gamma_\lambda = 2\Delta + \epsilon(1-\alpha) \tag{6.34}$$

$$\Omega_1 = 2\epsilon, \quad \Omega_2 = -\eta + (1-\alpha)\epsilon, \quad \Omega_3 = -2\Delta + (2\alpha - 1)\epsilon,$$

$$\beta = \frac{2+2\Delta}{2-(1-\alpha)\epsilon},$$

The nontrivial case is obtained when no special choice for parameter u is considered, in contrary let's consider u finite and non-zero. From (5.18) we deduce than g also has to be non-zero. Solving equations $\beta_g = \beta_u = 0$ for the case $u \neq 0, g \neq 0$ we obtain following values of fixed point

$$\frac{g^*}{8\pi u^*(1+u^*)} = \frac{2\epsilon - \eta}{1+\alpha}, \quad u^* = -1 + \frac{\alpha(\eta - 2\epsilon)}{(1+\alpha)(\eta - \epsilon)}\,. \tag{6.35}$$

The two regimes with this coordinates are distinguished by the value of coordinate λ^*. Fixed point with zero value can be characterised as follows

$$\textbf{FP 9:} \qquad \begin{aligned} &\lambda^* = 0\,, \\ &\gamma_D = \eta, \quad \gamma_v = \eta - \epsilon, \quad \gamma_\lambda = \frac{\alpha(2\epsilon - \eta)}{1+\alpha}\,, \\ &\beta = \frac{2 - \eta + \alpha(1 + \epsilon - \eta)}{(1+\alpha)(2-\eta)}\,, \end{aligned} \tag{6.36}$$

that is stable for the region

$$(1-\alpha)\epsilon < \eta < \epsilon\,, \tag{6.37}$$

$$\Delta + \frac{(1+2\alpha)\eta}{2(1+\alpha)} > \frac{\alpha\epsilon}{1+\alpha}\,. \tag{6.38}$$

The fixed point with non-zero value of λ^* is given as

$$\textbf{FP 10:} \qquad \begin{aligned} &\lambda^* = -4\pi\Delta + \frac{2\pi\alpha\epsilon}{1+\alpha} - 2\pi\eta\frac{1+2\alpha}{1+\alpha}\,, \\ &\gamma_D = \eta, \quad \gamma_v = \eta - \epsilon, \quad \gamma_\lambda = 2\Delta + \eta\,, \\ &\beta = \frac{2 + 2\Delta}{2 - \eta}\,, \end{aligned} \tag{6.39}$$

and it is stable for the region

$$(1-\alpha)\epsilon < \eta < \epsilon, \tag{6.40}$$

$$\Delta + \frac{(1+2\alpha)\eta}{2(1+\alpha)} < \frac{\alpha\epsilon}{1+\alpha}\,. \tag{6.41}$$

The "real problem" corresponds to the value $\epsilon = 4/3$, which leads to the famous Kolmogorov "five-thirds law" [15] for the spatial velocity statistics.

The phase diagrams in (ϵ, Δ)-plane ($\eta = 0$) and in ϵ, η-plane ($\Delta = 0$) are depicted in Figures 13 and 14. The broad lines represents boundaries between given regimes.

6.5 CONCLUSIONS

Fixed points **1** and **5** are not very interesting from the physical point of view. They corresponds to the non-interacting (mean-field) theory. Actually their presence is needed for the correct use of RG method [10].

From the fixed point structure some physical consequences can be deduced. First we see, that compressibility has direct influence on the value of decaying exponent (see **FP 3**, **FP 7-9**). It could lead (**FP 3**, **FP 7**) to the enhancement of the reaction process than

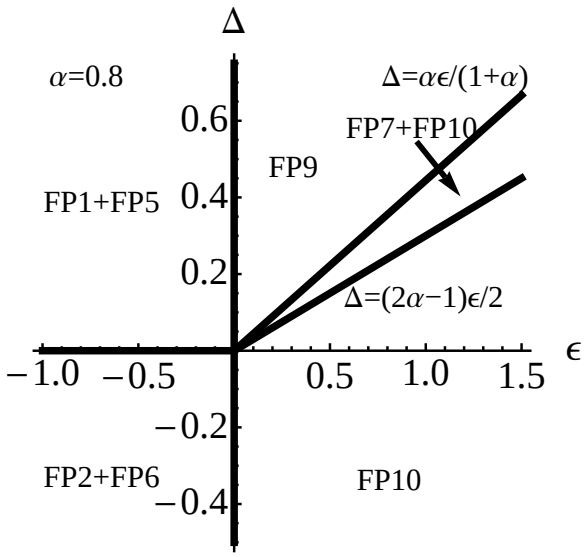


Figure 13: Phase diagram in the (ϵ, Δ) plane for $\eta = 0$ (no deviations from the parabolic law for dispersion relation)

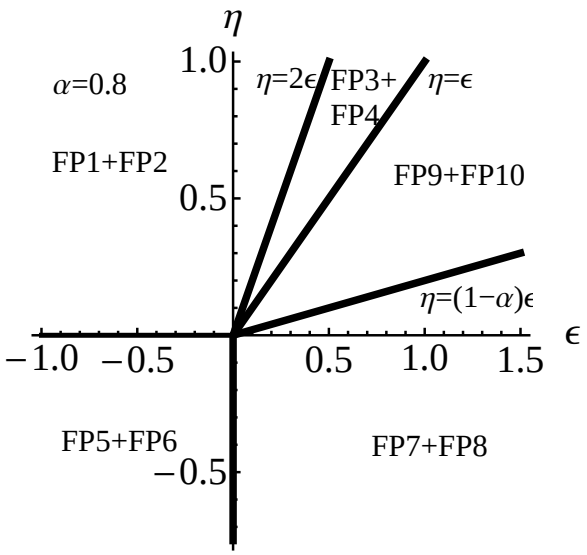


Figure 14: Phase diagram in the (ϵ, η) plane for $\Delta = 0$ ($d = 2$)

the corresponding regimes for incompressible case (see Tab. 5). It can be explained by the presence of compressible sinks into which particles are attracted. However we also see that when both density fluctuations and compressiblity are relevant (**FP 4**, **FP 8**, **FP 10**), the density tends to suppres effect of compressiblity.

7 ROLE OF RANDOM SOURCES AND SINKS ON REACTION PROCESSES

7.1 INTRODUCTION

The result (3.59) reminds field-theoretic models of critical dynamics obtained from the Langevin equation [1] within the Martin-Siggia-Rose approach. In the Langevin equation approach, the random field compensates for dissipative and reactive losses hence bringing about a steady state of dynamics of the system. In reaction kinetics the random sources and sinks, in fact, reflect the real physical situation, in which during the chemical reaction of follow-up species, particles can appear or disappear due to uncontrolled random interaction with a particle bath. Possible example can be the situation in atmosphere, where hydrogen molecule can undergo reaction scheme $H_2 \leftrightarrows H + H$, where H is hydrogen atom.

The interpretation of random fields in the Langevin equation as physical sources and sinks is somewhat problematic. Therefore, we propose to analyze the alternative approach provided by the master equation with terms corresponding to interactions with the bath. In the present paper we follow the ideas of [13] and describe the random sources and sinks in terms of new birth and death reactions in the master equation for the single-species annihilation reaction. The simplest choice does not conserve the particle number and we have no possibility to compare it with the standard Langevin approach. A slightly more involved set of birth-death reactions allows to conserve the particle number and the result may be compared to the Langevin equation with the standard multiplicative noise. This section is organized as follows. In Section 7.2 we formulate general formalism for the master equation for the annihilation reaction with random sources and sinks. We recall the basic features of the Doi formalism and construct the basic field-theoretic dynamic action functional. Scaling analysis of the dynamic actions is presented in Section 7.3, whereas Section 7.4 is devoted to discussion and concluding remarks.

7.2 MASTER EQUATION FOR RANDOM SOURCES AND SINKS

We will consider the annihilation reaction $A + A \to \emptyset$ in a random drift field in a more general setup than in the previous parts. For this purpose we introduce random sources and sinks of the reacting particles in order to maintain a steady state in the system. In most cases this is carried out by including an additive noise term in the Langevin equation of the stochastic process as was done e.g. in (2.1) to have steady turbulent state. Since our analysis is based on the master equation, this is not quite appropriate here.

Unfortunately, there is no unique way to introduce random sources in the master equation corresponding to the random noise of the mean-field (Langevin) description. We use the simplest choice, described in detail in [13], which is equivalent to adding processes $A \to X$ and $Y \to A$ .to the whole reaction scheme. Here X and Y stand for particle baths of the sink and the source respectively. In a homogeneous system these reactions leads to the master equation

$$\frac{\mathrm{d}P(t,n)}{\mathrm{d}t} = \mu_+ V\,[P(t,n-1)-P(t,n)]+\mu_-\,[(n+1)P(t,n+1)-nP(t,n)]\ldots \qquad (7.1)$$

where $P(t,n)$ is the probability to find n particles at the time instant t in the system. The ellipsis in (7.1) represents terms describing the annihilation reaction, diffusion and advection in the system. In (7.1) μ_+ and μ_- are the reaction constants of the creation and annihilation reactions, respectively. The transition rate has been chosen proportional to the particle number n, which can be understood as consequence of independent processes $A \to X$ and this choice also preserves the empty state as an absorbing state. In the transition rate for creation process V is the volume of the (for the time being) homogeneous system and will be important in passing to the continuum limit of the inhomogeneous system. The master equation (7.1) gives rise to the reaction-rate equation

$$\frac{\mathrm{d}\langle n\rangle}{\mathrm{d}t} = \mu_+ V - \mu_-\langle n\rangle + \ldots \qquad (7.2)$$

where $\langle n\rangle$ is the mean particle number.

We recall that the basice idea of the Doi approach [26] is to rewrite the set of master equations for probability distributions of a stochastic problem in the form of a single kinetic equation for a state vector incorporating all probabilistic information about the system constructed in a suitable Fock space. The kinetic equation is defined by the Liouville operator acting in the Fock space and generated by the set of master equations. Although the basic procedure has been thoroughly exposed in the literature, the introduction of random sources and sinks of particles in the master equation has specific features, which should be presented in detail. Therefore, let us briefly recall the basic quantities and relations of the Doi approach. For simplicity, consider probabilities $P(t,n)$ to find n particles at the time instant t on a fixed lattice site. Then the spatial dependence may be described by labeling the particle number by the coordinates of the lattice and introducing necessary sums and products over the lattice sites. The construction of corresponding Fock space was presented in Section. 3.2, namely equations (3.7-3.14). The set of master equations

for a birth-death process may also be cast in the form of a single evolution equation for the state vector (3.14) without any explicit dependence on the occupation number

$$\frac{\mathrm{d}|\Phi\rangle}{\mathrm{d}t} = \hat{L}(\hat{a}^+, \hat{a})|\Phi\rangle\,. \tag{7.3}$$

Master equations (7.1) give rise to the following terms in the Liouville operator

$$\hat{L}_g(\hat{a}^+, \hat{a}) = \mu_+ V\left(\hat{a}^+ - I\right) + \mu_-\left(I - \hat{a}^+\right)\hat{a}\,, \tag{7.4}$$

where I is the identity operator. The expectation value of any function $F(n)$ of the random particle number

$$\langle F(t)\rangle = \sum_{n=0}^{\infty} F(n)P(t,n), \tag{7.5}$$

may be expressed in the form of the functional integral over the functions $\tilde{a}(t)$ and $a(t)$

$$\langle F(t)\rangle = \int \mathscr{D}\tilde{a}\mathscr{D}a\, F_N[(\tilde{a}(t)+1)a(t)]e^{S_1}\,, \tag{7.6}$$

where $F_N(\tilde{a}a)$ is the normal form [32] of the operator $F(\hat{a}^+\hat{a})$ and S_1 is the dynamic action

$$S_1(\tilde{a}, a) = \int_0^{\infty} dt\,\left[-\tilde{a}(t)\partial_t a(t) + \mu_+ V\tilde{a}(t) - \mu_-\tilde{a}(t)a(t)\right]\ldots \tag{7.7}$$

Only the generic time-derivative term and terms brought about by the random source model are expressed here explicitly, while the ellipsis stands for terms corresponding to other reactions and initial conditions. Let the transition rates $\mu_\pm$ be the random functions uncorrelated in time with a probability distribution given in terms of the moments $\langle\mu_\pm^n\rangle = E_{\pm,n}$. At this point we also generalize the treatment to the case of a spatially inhomogeneous system and introduce a lattice subscript as the spatial argument, i.e. $a(t) \to a_i(t)$. In this case the volume V becomes the volume element attached to the lattice site. For simplicity, we replace the time integral with the integral sum $\int_0^\infty dt \to \sum_\alpha \Delta t$ and assume that the transition rates at each time instant and lattice site $\mu_{\pm,\alpha,i}$ are independent random variables. Then the average of the expectation value (7.6) over the distribution of random sources reduces to the calculation of the expectation value

$$\prod_{\alpha,i}\left\langle \exp\left(\mu_{+,\alpha,i} V\tilde{a}_{\alpha,i}\Delta t - \mu_{-,\alpha,i}\tilde{a}_{\alpha,i}a_{\alpha,i}\Delta t\right)\right\rangle. \tag{7.8}$$

For each particular time instant and lattice (we assume that the moments of $\mu_\pm$ are the

same for all α and i and omit labels for brevity) this gives rise to the usual cumulant expansion

$$\begin{aligned}\left\langle \exp\left(\mu b \Delta t\right)\right\rangle &= 1 + b\Delta t E_1 + \frac{1}{2}E_2(b\Delta t)^2 + \frac{1}{6}E_3(b\Delta t)^3 + \cdots \\ &= \exp\left(b\Delta t E_1 + \frac{E_2 - E_1^2}{2}(b\Delta t)^2 + \frac{E_3 - 3E_1E_2 + E_1^3}{6}(b\Delta t)^3 + \cdots\right) \quad (7.9)\end{aligned}$$

Here, b stands for either $V\tilde{a}$ or $-\tilde{a}a$. Thus, for instance the average over μ_+ assumes the form

$$\begin{aligned}\prod_{\alpha,i}\left\langle \exp\left(\mu_{+,\alpha,i} V\tilde{a}_{\alpha,i}\Delta t\right)\right\rangle &= \exp\left(\sum_\alpha\sum_i\left[\Delta t E_{+1} V\tilde{a}_{\alpha,i} + \frac{1}{2}\left(E_{+2} - E_{+1}^2\right)(V\tilde{a}_{\alpha,i}\Delta t)^2\right]\right) \\ &\times \exp\left(\sum_\alpha\sum_i\left[\frac{E_{+3} - 3E_{+1}E_{+2} + E_{+1}^3}{6}(V\tilde{a}_{\alpha,i}\Delta t)^3 + \cdots\right]\right).\end{aligned} \quad (7.10)$$

In the continuum limit the function $\tilde{a}_{\alpha,i}$ is replaced by the field $\psi^+(t,\boldsymbol{x})$, whereas in the limit $V \to 0$ the expression $a_{\alpha,i}/V$ gives rise to the field $\psi(t,\boldsymbol{x})$. The sum over α together with Δt gives rise to the time integral and the sum over i together with the volume element leads to the spatial integral $\sum_i V \to \int d\boldsymbol{x}$. In the first term of the exponential in (7.10) we thus obtain

$$\sum_\alpha\sum_i \Delta t E_{+1} V\tilde{a}_{\alpha,i} \to E_{+1}\int dt\int d\boldsymbol{x}\,\psi^+(t,\boldsymbol{x}). \quad (7.11)$$

The continuum limit for the cumulants of second and higher order is not so obvious. We assume the simplest nontrivial distribution for $\mu_\pm$, in which only the variance term has a finite limit, when $\Delta t \to 0$ and $V \to 0$, whereas the contributions of higher-order cumulants vanish, for instance

$$\left(E_{+2} - E_{+1}^2\right)V\Delta t \to \sigma_+, \quad \Delta t \to 0,\ V \to 0, \quad (7.12)$$

$$\left(E_{+3} - 3E_{+1}E_{+2} + E_{+1}^3\right)(V\Delta t)^2 \to 0, \quad \Delta t \to 0,\ V \to 0. \quad (7.13)$$

Therefore, the contribution of the average over μ_+ to the effective dynamic action assumes the form

$$S_+ = \int dt\int d\boldsymbol{x}\left\{E_{+1}\psi^+(t,\boldsymbol{x}) + \frac{1}{2}\sigma_+\left[\psi^+(t,\boldsymbol{x})\right]^2\right\}. \quad (7.14)$$

For the average over μ_- a similar argument yields

$$S_- = \int dt \int dx \left\{ -E_{-1}\psi^+(t,x)\psi(t,x) + \frac{1}{2}\sigma_- \left[\psi^+(t,x)\psi(t,x)\right]^2 \right\}. \tag{7.15}$$

These contributions to the effective dynamic action may, of course, be generated by suitably chosen normal distributions of $\mu_\pm$.

This way of introduction of random sources and sinks has the annoying feature that it does not conserve the number of particles in the system. For a comparison with the treatment of this problem in the Langevin approach the random sources and sinks should be introduced in such a way that the particle number is conserved. The simplest way how to deal with this problem is to add to the random source a term proportional to the particle number, i.e. to use the "reaction constant" $\mu_+ V + \mu_{1+} n$ instead of $\mu_+ V$ in the master equation. The source terms on the right-hand side of the master equation (7.1) in this case assume the form

$$\begin{aligned} \frac{\mathrm{d}P(t,n)}{\mathrm{d}t} &= \mu_+ V \left[P(t,n-1) - P(t,n)\right] \\ &+ \mu_{1+} \left[(n-1)P(t,n-1) - nP(t,n)\right] \ldots \end{aligned} \tag{7.16}$$

The new part of the master equation corresponds to a branching process [13].

The added term gives rise to the following contribution to the Liouville operator

$$\hat{L}_{g2}(\hat{a}^+, \hat{a}) = \mu_{1+} \left(\hat{a}^+ - I\right) \hat{a}^+ \hat{a}\,. \tag{7.17}$$

Performing the steps described above we arrive at the contribution to the dynamic action in the following form

$$S_{1+} = \int dt \int dx \left\{ E_{1+1}\psi^+ \left(\psi^+ + 1\right)\psi + \frac{1}{2}\sigma_{1+}\psi^{+2}\left(\psi^+ + 1\right)^2 \psi^2 \right\}. \tag{7.18}$$

Now it is easy to see, that if we exclude the plain source (i.e. letting $E_{+1} = \sigma_+ = 0$) and choose $E_{1+1} = E_{-1}$, the empty state remains absorbing one and the "mass term" $\propto \psi^+\psi$ disappears in the dynamic action and we arrive at the dynamic action of random sources and sinks

$$S_{gc} = \int dt \int dx \left\{ E_{1+1}\psi^{+2}\psi + \frac{1}{2}\sigma_- \left(\psi^+\psi\right)^2 + \frac{1}{2}\sigma_{1+}\psi^{+2}\left(\psi^+ + 1\right)^2 \psi^2 \right\}, \tag{7.19}$$

which conserves the average number of particles.

The effects of the high-order terms are drastically different in the two cases amenable for a scaling analysis with the aid of the renormalization group. The time derivative term in the dynamic action

$$S = -\int dt \int d\boldsymbol{x}\, \psi^{+}(t,\boldsymbol{x})\partial_t \psi(t,\boldsymbol{x}) + \ldots \tag{7.20}$$

must be dimensionless in order to have nontrivial dynamics. Therefore the total scaling dimension of the number-density operator $\psi^{+}(t,\boldsymbol{x})\psi(t,\boldsymbol{x})$ is equal to the dimension of space and thus is positive.

First, if the scaling dimension of the field ψ^{+} is equal to zero, $d_{\psi^{+}} = 0$, then the dimension of the field ψ is positive (more precisely $d_{\psi} = d$) and the operator monomials in the second and third terms in (7.19) have the same scaling dimension. Since they are carrying the factor ψ^2, their scaling dimension is larger than that of $\psi^{+2}\psi$. Therefore, the second and third terms in (7.19) are IR irrelevant and should be discarded in the asymptotic analysis.

Second, if the scaling dimensions of both fields are positive, then in the operator monomials in the second and third terms in (7.19) there is at least one "excessive" field factor in comparison with the first term, which renders them irrelevant. Thus, in these cases the IR relevant dynamic action of random sources and sinks reduces to the single term

$$S'_{gc} = \int dt \int d\boldsymbol{x} E_{1+1} \psi^{+2}\psi, \quad d_{\psi^{+}} = 0 \ \vee \ d_{\psi^{+}} > 0,\ d_{\psi} > 0. \tag{7.21}$$

Third, if the scaling dimension of the field ψ is zero, the scaling dimension of the field ψ^{+} is positive and terms with "excessive" powers of ψ^{+} are IR irrelevant. So the starting point for the subsequent RG analysis is the source and sink action in the form

$$S''_{gc} = \int dt \int d\boldsymbol{x} \left\{ E_{1+1}\psi^{+2}\psi + \frac{1}{2}(\sigma_{-} + \sigma_{1+})\left(\psi^{+}\psi\right)^2 \right\}, \quad d_{\psi} = 0. \tag{7.22}$$

7.3 ANNIHILATION PROCESS WITH RANDOM SOURCES AND SINKS

Let us analyze the dynamic action of the diffusion-limited annihilation reaction $A + A \to \emptyset$

$$S_1 = -\int_0^{\infty} dt \int d\mathbf{x} \left\{ \psi^{+}\partial_t \psi - D_0 \psi^{+}\nabla^2 \psi + \lambda_0 D_0 \left[2\psi^{+} + (\psi^{+})^2 \right] \psi^2 \right\} + n_0 \int d\mathbf{x}\, \psi^{+}(\mathbf{x}, 0), \tag{7.23}$$

from the point of view of scaling behavior sketched in section 7.2.

In the first case with $d_{\psi^+} = 0$ the nonlinear terms in action (7.23) are of equal scaling dimension. However, the source-sink part (7.21) is linear in the field ψ with positive scaling dimension in contrast to the quadratic ψ terms of (7.23). Therefore, the IR relevant interaction above two dimensions is (7.21) and the corresponding dynamic action is

$$S_{IR1} = -\int_0^\infty dt \int d\mathbf{x} \left\{ \psi^+ \partial_t \psi - D_0 \psi^+ \nabla^2 \psi - E_{1+1} \psi^{+2} \psi \right\} + n_0 \int d\mathbf{x} \psi^+(\mathbf{x}, 0). \quad (7.24)$$

This dynamic action does not bring about any graphs with closed loops of the density propagator, which implies suppresion of the fluctuation effects. However, the scaling dimension of the interaction term is negative and may compensate for the positive dimensions of the irrelevant interaction terms. Therefore, the rest of the interaction terms are in fact dangerous irrelevant operators and in this case definitive conclusion about the IR relevant action cannot be reached on the basis of the analysis of the scaling dimensions.

In the second case with $d_{\psi^+} > 0$ and $d_\psi > 0$ the fourth-order term in action (7.23) becomes irrelevant. Either of the remaining third-order terms alone does not generate loops, therefore density fluctuation effects are brought about only, when both fields have the same scaling dimension $d_{\psi^+} = d_\psi = d/2$. In this case the IR relevant dynamic action is

$$S_{IR2} = -\int_0^\infty dt \int d\mathbf{x} \left\{ \psi^+ \partial_t \psi - D_0 \psi^+ \nabla^2 \psi + 2\lambda_0 D_0 \psi^+ \psi^2 - E_{1+1} \psi^{+2} \psi \right\} + n_0 \int d\mathbf{x} \psi^+(\mathbf{x}, 0). \quad (7.25)$$

Here, the scaling dimension of both interaction terms is $(d/2) - 2$ and vanishes at the critical dimension $d_c = 4$, at which the dimensions of all the other interaction terms are positive and they are unambiguously irrelevant. Effective action (7.25) is the dynamic action of the Gribov process [78], also known as the Reggeon model. Effects of random drift in this case with the use of the Obukhov-Kraichnan compressible velocity field have been analyzed in [99].

In the third case with $d_\psi = 0$ the fourth-order term in action (7.23) becomes irrelevant as well due to the positive dimension of the field ψ^+. By the same token, however, both terms of the source-sink action (7.22) are also irrelevant and we arrive at the IR relevant dynamic action

$$S_{IR3} = -\int_0^\infty dt \int d\mathbf{x} \left\{ \psi^+ \partial_t \psi - D_0 \psi^+ \nabla^2 \psi + 2\lambda_0 D_0 \psi^+ \psi^2 \right\} + n_0 \int d\mathbf{x} \psi^+(\mathbf{x}, 0). \quad (7.26)$$

An argument similar to that used for (7.24) shows, that the scaling analysis with this choice of field dimensions does not allow to resolve relevance of interaction terms. It should be recalled that the scaling dimensions of auxiliary quantities and the asymptotic behavior of individual graphs is actually independent of the choice of the values of the field dimensions. Therefore, the effective action (7.25) with unambiguous classification of relevant and irrelevant interaction terms describes the critical scaling behavior amenable to the RG analysis.

In summary, if the sources and sinks are chosen such that they conserve the mean number of particles in the system, the anomalous scaling behavior in the system is that of the Gribov process.

A different situation arises, if the plain source term is included into the analysis. Then there is a possibility that the system does not tend to the absorbing empty state but to an active state with a finite concentration of particles. In this case the starting point is the dynamic action with all the terms quoted above, i.e.

$$\begin{aligned} S &= \int_0^\infty dt \int d\mathbf{x} \Big\{ -\psi^+ \partial_t \psi + D_0 \psi^+ \nabla^2 \psi - \lambda_0 D_0 \left[2\psi^+ + \left(\psi^+\right)^2 \right] \psi^2 + E_{+1} \psi^+ \\ &+ \frac{1}{2} \sigma_+ \left(\psi^+\right)^2 + E_{1+1} \psi^+ \left(\psi^+ + 1\right) \psi + \frac{1}{2} \sigma_{1+} \psi^{+2} \left(\psi^+ + 1\right)^2 \psi^2 - E_{-1} \psi^+ \psi \\ &+ \frac{1}{2} \sigma_- \left(\psi^+ \psi\right)^2 \Big\} + n_0 \int d\mathbf{x} \, \psi^+(\mathbf{x}, 0) . \end{aligned} \tag{7.27}$$

The stationarity equation brought about by this dynamic action for the field ψ is (the stationary value $\psi^+ = 0$ as usual)

$$\partial_t \psi - D_0 \nabla^2 \psi = -2\lambda_0 D_0 \psi^2 + E_{+1} + E_{1+1} \psi - E_{-1} \psi . \tag{7.28}$$

However, the action expanded around the stationary value is rather complicated. To keep expressions simple, continue to consider the case $E_{1+1} = E_{-1}$. Then the re-expanded ac-

tion is

$$
\begin{aligned}
S &= \int_0^\infty dt \int d\mathbf{x} \Big\{ -\psi^+ \partial_t \psi + D_0 \psi^+ \nabla^2 \psi - \sqrt{8} \sqrt{E_{+1} \lambda_0 D_0} \psi^+ \psi \\
&+ \left(\frac{-E_{+1}}{2} + \frac{E_{-1} \sqrt{E_{+1} \lambda_0 D_0}}{\sqrt{2} \lambda_0 D_0} + \frac{E_{+1} \sigma_{1+}}{4 \lambda_0 D_0} + \frac{E_{+1} \sigma_-}{4 \lambda_0 D_0} + \frac{\sigma_+}{2} \right) \psi^{+2} \\
&+ \frac{E_{+1} \sigma_{1+} \psi^{+3}}{2 \lambda_0 D_0} + \frac{E_{+1} \sigma_{1+} \psi^{+4}}{4 \lambda_0 D_0} + \frac{\sqrt{2} \sqrt{E_{+1} \lambda_0 D_0} \sigma_{1+} \psi^{+3} \psi}{\lambda_0 D_0} \\
&+ \left(E_{-1} - \sqrt{2} \sqrt{E_{+1} \lambda_0 D_0} + \frac{\sqrt{E_{+1} \lambda_0 D_0} \sigma_{1+}}{\sqrt{2} \lambda_0 D_0} + \frac{\sqrt{E_{+1} \lambda_0 D_0} \sigma_-}{\sqrt{2} \lambda_0 D_0} \right) \psi^{+2} \psi \\
&+ \frac{\sqrt{E_{+1} \lambda_0 D_0} \sigma_{1+} \psi^{+4} \psi}{\sqrt{2} \lambda_0 D_0} - 2 \lambda_0 D_0 \psi^+ \psi^2 + \left(-\lambda_0 D_0 + \frac{\sigma_{1+}}{2} + \frac{\sigma_-}{2} \right) \psi^{+2} \psi^2 \\
&+ \sigma_{1+} \psi^{+3} \psi^2 + \frac{\sigma_{1+} \psi^{+4} \psi^2}{2} \Big\} + n_0 \int d\mathbf{x} \, \psi^+(\mathbf{x}, 0) .
\end{aligned} \tag{7.29}
$$

In the critical limit $E_{+1} \to 0$. Since it is the expectation value of a nonnegative random quantity μ_+, the variance σ_+ vanishes as well. In the vicinity of the critical point we keep only the leading E_{+1} and σ_+ putting them equal zero in terms, where they are subleading. This simplifies the action a little bit

$$
\begin{aligned}
S &= \int_0^\infty dt \int d\mathbf{x} \Big\{ -\psi^+ \partial_t \psi + D_0 \psi^+ \nabla^2 \psi - \sqrt{8} \sqrt{E_{+1} \lambda_0 D_0} \psi^+ \psi + \left(\frac{E_{-1} \sqrt{E_{+1}}}{\sqrt{2 \lambda_0 D_0}} + \frac{\sigma_+}{2} \right) \psi^{+2} \\
&+ \frac{E_{+1} \sigma_{1+} \psi^{+3}}{2 \lambda_0 D_0} + \frac{E_{+1} \sigma_{1+} \psi^{+4}}{4 \lambda_0 D_0} + E_{-1} \psi^{+2} \psi + \frac{\sqrt{2} \sqrt{E_{+1} \lambda_0 D_0} \sigma_{1+} \psi^{+3} \psi}{\lambda_0 D_0} \\
&+ \frac{\sqrt{E_{+1} \lambda_0 D_0} \sigma_{1+} \psi^{+4} \psi}{\sqrt{2} \lambda_0 D_0} - 2 \lambda_0 D_0 \psi^+ \psi^2 + \left(-\lambda_0 D_0 + \frac{\sigma_{1+}}{2} + \frac{\sigma_-}{2} \right) \psi^{+2} \psi^2 \\
&+ \sigma_{1+} \psi^{+3} \psi^2 + \frac{\sigma_{1+} \psi^{+4} \psi^2}{2} \Big\} + n_0 \int d\mathbf{x} \, \psi^+(\mathbf{x}, 0) .
\end{aligned} \tag{7.30}
$$

Dimensional analysis of the canonical dimensions then yields the following cases. In the nonlinear parts without the critical parameters E_{+1} and σ_+ the previous arguments hold, but in terms having powers of these parameters as coefficients the positive scaling dimensions of them must be taken into account. The free-field part of the action (7.30) suggests that the canonical dimension of E_{+1} is four. In fact, the canonical dimension of σ_+ remains a free parameter.

Proceeding in the same manner as above, we arrive at the following effective actions for the IR scaling limit. In the first case with $d_{\psi^+} = 0$ the third and fourth powers of ψ^+

and independent of ψ or first order in ψ are irrelevant (due to the coefficients proportional to E_{+1} or its square root) compared with terms $\propto \psi^{+2}$ in action (7.30). Nonlinear in ψ terms are irrelevant against the linear terms due to the positive dimension of ψ. Therefore, the IR effective action in this case is

$$\begin{aligned} S &= \int_0^\infty dt \int d\mathbf{x} \Big\{ -\psi^+ \partial_t \psi + D_0 \psi^+ \nabla^2 \psi - 2\sqrt{2}\sqrt{E_{+1}\lambda_0 D_0}\,\psi^+\psi \\ &+ \left(\frac{E_{-1}\sqrt{E_{+1}}}{\sqrt{2\lambda_0 D_0}} + \frac{\sigma_+}{2} \right) \psi^{+2} + E_{-1}\psi^{+2}\psi \Big\} + n_0 \int d\mathbf{x}\, \psi^+(\mathbf{x},0). \end{aligned} \tag{7.31}$$

Again, the interaction term remaining after the formal dimensional analysis does not bring about loops, although here we have a nontrivial correlation function of the field ψ. The scaling dimension of this term is negative, however, rendering the irrelevant terms dangerous and prohibiting any definitive conclusion about the relevance of individual interaction terms.

In the second case with $d_{\psi^+} > 0$ and $d_\psi > 0$ higher powers than the leading corrections to the free-field action of both fields are irrelevant. This argument leaves us with the dynamic action

$$\begin{aligned} S &= \int_0^\infty dt \int d\mathbf{x} \Big\{ -\psi^+ \partial_t \psi + D_0 \psi^+ \nabla^2 \psi - 2\sqrt{2}\sqrt{E_{+1}\lambda_0 D_0}\,\psi^+\psi \\ &+ \left(\frac{E_{-1}\sqrt{E_{+1}}}{\sqrt{2\lambda_0 D_0}} + \frac{\sigma_+}{2} \right) \psi^{+2} + E_{-1}\psi^{+2}\psi - 2\lambda_0 D_0 \psi^+\psi^2 \Big\} + n_0 \int d\mathbf{x}\, \psi^+(\mathbf{x},0). \end{aligned} \tag{7.32}$$

Contrary to the case discussed above, here the interaction term $-2\lambda_0 D_0 \psi^+\psi^2$ generates loops alone due to the presence of the correlation function of the field ψ. Therefore, two effective actions with nontrivial fluctuation contributions are possible.

a) $d_{\psi^+} > d_\psi$. To keep the correlation function of the field ψ for the loops, the variance σ_+ must have a dimension less than that of $\sqrt{E_{+1}}$. This yields the effective action

$$\begin{aligned} S &= \int_0^\infty dt \int d\mathbf{x} \Big\{ -\psi^+ \partial_t \psi + D_0 \psi^+ \nabla^2 \psi - 2\sqrt{2}\sqrt{E_{+1}\lambda_0 D_0}\,\psi^+\psi \\ &+ \frac{\sigma_+}{2} \psi^{+2} - 2\lambda_0 D_0 \psi^+\psi^2 \Big\} + n_0 \int d\mathbf{x}\, \psi^+(\mathbf{x},0) \end{aligned} \tag{7.33}$$

with the critical dimension depending on the scaling dimension of σ_+ in the spirit of the description of tricritical scaling behavior [1]. The model is logarithmic at six dimensions, however, because apart from the coefficient of the $\propto \psi^{+2}$ the action is that of critical dynamics of the φ^3 model. However, also in this case some irrelevant operators are po-

tentially dangerous and the asymptotic behavior of the model is, in fact, described by the following effective action.

b) $d_{\psi^+} = d_\psi = d/2$. Both third-order terms are relevant and the effective action is basically (7.32). In this case the dimension of σ_+ is larger than that of $\sqrt{E_{+1}}$ and for simplicity we omit σ_+. Thus, the effective dynamic action may be written as

$$\begin{aligned} S &= \int_0^\infty dt \int d\mathbf{x} \Big\{ -\psi^+ \partial_t \psi + D_0 \psi^+ \nabla^2 \psi - 2\sqrt{2}\sqrt{E_{+1}\lambda_0 D_0}\,\psi^+\psi \\ &+ \frac{E_{-1}\sqrt{E_{+1}}}{\sqrt{2\lambda_0 D_0}}\psi^{+2} + E_{-1}\psi^{+2}\psi - 2\lambda_0 D_0 \psi^+ \psi^2 \Big\} + n_0 \int d\mathbf{x}\, \psi^+(\mathbf{x},0). \end{aligned} \tag{7.34}$$

Note that this is a dynamic action describing the Gribov process with a random source independent of the active agent density. That the rate of change of the density due to the random sink is proportional to a power of density is a natural assumption. The assumption that the rate of change of the density due to the random source is proportional to a power of density is not natural. Therefore, the dynamic action (7.34) possibly predicts a critical behavior of the Gribov process different from that discussed in the literature.

In the third case with $d_{\psi^+} > 0$ and $d_\psi = 0$ we arrive at the effective action (7.33).

The analysis of scaling dimensions shows that we may actually lift most of the restrictions on the probability distribution of the transition rates of the type (7.12) and (7.13). Indeed, even if the higher order cumulants are finite, the scaling dimensions of corresponding terms in the dynamic action grow with the order of the cumulant with the exception of the case, when the transition rate is independent of the agent density.

7.4 CONCLUSION

We have investigated possible effects of random sources and sinks on the pair annihilation reaction $A + A \to \emptyset$. Contrary to the frequently used approach, in which the sources and sinks are introduced into the Langevin equation, we have included them directly to the master equation, where their physical sense is clear. We have considered linear in particle number creation and annihilation reactions with random rate coefficients to model the sources and sinks. On the basis of the analysis of canonical scaling dimensions we have constructed effective actions, which are the starting point for an RG analysis of the critical behavior of the systems under consideration. In all cases the effect of random sources and sinks to the large-scale, long-time behavior of the Green functions is significant and changes the universality class of the model. Instead of the universality class of the pair annihilation reaction $A + A \to \emptyset$, the asymptotic behavior of the model

with random sources and sinks belongs to the universality class of the Gribov process in the critical case and to a modified Gribov process in the critical limit of the noncritical model. In the former case it is demonstrated once again that the description of a stochastic process with the use of the Langevin equation is significantly different from the description in terms of a master equation. The random noise term in the Langevin equation corresponds rather to the account of effects of genuine random sources and sinks than to a description of the effect of microscopic degrees of freedom on the mesoscopic process. Here, the universality classes of the same reaction process are completely different in the case of the master equation without sources and sinks in comparison to the case of Langevin equation for the same process.

In the noncritical case with a random source independent of the agent density the Gribov process is modified to account for effects in critical behavior, when sources and sinks asymptotically vanish. The analysis of the dependence of scaling functions on the parameters of the probability distribution of sinks and sources in infrared limit is called for. This reminds the situation, which takes place in the theory of phase transitions, where statistical correlations of the order parameter depend on a "mass" (deviation of temperature from the critical value) and the dependence of the scaling functions on the "‘mass" is investigated.

8 DIRECTED PERCOLATION IN THE PRESENCE OF SYNTHETIC VELOCITY FIELD

8.1 INTRODUCTION

Directed bond percolation (DP) problem is prominent model of statistical physics. In different formulations it can serve for explaining hadron interactions at very high energies (Reggeon field theory) [79, 80], various models of disease spreading [81, 82] or as in original formulation [83] wetting of porous material or exploring path in labyrinth. In the statistical sense it is also the simplest model, that has second-order phase transition [81] between absorbing and fluctuating phases. The upper critical dimension for this problem was estimated to be $d = 4$ in contrast to the value $d = 6$ for the isotropic dynamical case [81, 84, 85], which we do not consider here.

It was conjectured [86, 87] that critical behavior of any one-component system with only short-range interactions with second-order transition to absorbing state belong always to the DP class. Despite the general validity of this statement the percolation model was not experimentalluy realized until recently [88]. In this work transitions between two topologically different states in electrohydrodynamic convection were observed. Diffuculties with direct experimental observation of percolation are caused by the presence of various inhomogenities, defects, anisotropies etc. and they could destroy or make observation of critical regime completely hopeless. Therefore it is important to study possible deviations from the simple percolation model and try to quantify their effects. A lot of effort was put into the investigation of effects as long-range interactions [89, 90, 94] by the means of Levy-flight jumps both in time and space variables, introduction of immunization [82, 91, 92], effect of surfaces [93] etc. One can also easily imagine that spreading of disease can be rapidly enhanced by some external atmospheric current or by some flying insects (as flee tse-tse). In both cases additional drift can be modelled by addition of random velocity field [95] with prescribed statistical properties. In this work only rapid-change Kraichnan model was studied (see definition 5.4). Generalizing this approach it is possible to study e.g. effect of compressibility [96] or using stochastic Navier-Stokes equations effects of "real" turbulent field [97]. In this section we investigate the influence of finite correlated velocity field (model used in Sections 5 and 6) and determine how it can change the critical behavior of percolation process. Using functional integral representation we apply field-theoretic renormalization group to determine possible universality classes. The model is analyzed near its critical dimension by means of three-parameter

expansion in ϵ, Δ, η, where ϵ is the deviation from the Kolmogorov scaling, Δ the deviation from the critical space dimension $d_c = 4$ and η is the deviation from the parabolic dispersion law for the velocity correlator. Fixed points with corresponding regions of stability are evaluated to the first order (one-loop) in the perturbation scheme. We determine possible large scale behavior and show that the model can exhibit one of 10 possible regimes. This section is organized as follows. In Section. 8.2 we give the basic description of the model. In Section 8.3 we present results of power counting for the model parameters and prove its multiplicative renormalizibility. In Section 8.4 we briefly describe main ingredients of the diagrammatic technique and present numerical results for the renormalization constants. In Section 8.5 we analyse asymptotic behavior of the model according to the fixed point structure and present range of stability in the (ϵ, Δ, η) space and finally Section 8.6 is devoted to conclusions and future plans.

8.2 FIELD-THEORETIC FORMULATION

Percolation process can be formulated as a reaction-diffusion process. According to the standard approach [81] master equation for such problem can be recast into the form of effective field-theoretic action, that is amenable to the usual field-theoretical methods. Contrary to the annihilation process other approach based on the use of Langevin equation with suitable chosen noise leads to the same prediction [81] of universal quantities as the master equation approach. Let us therefore briefly describe the main points of the latter. For concretness let us speak about disease, that is spreaded by the advecting field. The stochastic non-linear differential equation for the coarse-grained density of infected individuals (agents) $\psi(t, \mathbf{x})$ can be written in the following form [81, 98]

$$\partial_t \psi(t, \mathbf{x}) = D_0(\nabla^2 - \tau_0)\psi(t, \mathbf{x}) - \frac{\lambda_0 D_0}{2}\psi^2(t, \mathbf{x}) + \zeta(t, \mathbf{x}), \tag{8.1}$$

where ∇^2 is Laplace operator, D_0 is a diffusion constant and λ_0 is a positive coupling constant. The parameter τ_0 measures the deviation from the threshold value for the infection probability. One can assume $\tau_0 \sim p_c - p$, where p_c is a critical probability for observing percolation (analogous to the deviation from critical temperature for equilibrium models). It is important to note, that the model has unique absorbing state with $\psi(t, \mathbf{x}) = 0$ (no sick agents) from which it cannot escape. The Gaussian short-ranged noise $\zeta(x)$ with zero mean accounts for the density fluctuations and has to respect the absorbing state

condition. This can be achieved by the following choice of correlator [81, 89, 95]

$$\langle\zeta(t,\mathbf{x})\zeta(t',\mathbf{x}')\rangle = D_0\lambda_0\psi(t,\mathbf{x})\delta(t-t')\delta(\mathbf{x}-\mathbf{x}'). \tag{8.2}$$

In order to apply renormalization group technique for the study of large scale behavior of the model it is convenient to recast it into the path integral representation. The Langevin equation (8.1) has the standard form of stochastic dynamic problem (2.5). By introducing Martin-Siggia-Rose [28] response field $\psi^\dagger(t,\mathbf{x})$ and integrating out the Gaussian noise it is possible to obtain action functional [28, 80] for the pure directed percolation problem (or Reggeon field theory respectively)

$$S_{10}(\psi^\dagger,\psi) \quad = \quad \psi^\dagger(-\partial_t + D_0\nabla^2 - D_0\tau_0)\psi + \frac{D_0\lambda_0}{2}[(\psi^\dagger)^2\psi - \psi^\dagger\psi^2], \tag{8.3}$$

that corresponds to the action (2.6). For convenience the required integrations over the space-time variables are not explicitly indicated, e.g. the second term means the following expression

$$\psi^\dagger\nabla^2\psi = \int dt \int d\mathbf{x}\ \psi^\dagger(t,\mathbf{x})\nabla^2\psi(t,\mathbf{x}). \tag{8.4}$$

The model (8.3) satisfies so-called rapidity reversal symmetry (in the language of Reggeon field theory)

$$\psi(t,\mathbf{x}) \to -\psi^\dagger(-t,\mathbf{x}), \quad \psi^\dagger(t,\mathbf{x}) \to -\psi(-t,\mathbf{x}), \tag{8.5}$$

and it should be respected by the renormalization transformation at least for the absorbing state.

We would like to study the influence of advective field on the spreading of agents ψ. The agents can be considered as passive scalar quantity [21] that is advected by the velocity field with no back influence on the velocity field itself with nontrivial interactions given by the cubic terms in (8.3). The inclusion of the velocity field $\mathbf{v}(t,\mathbf{x})$ corresponds to the replacement

$$\partial_t \to \nabla_t = \partial_t + (\mathbf{v}.\boldsymbol{\nabla}), \tag{8.6}$$

similarly as was done for the annihilation process, see (4.2). For the generation of velocity fluctuations it is possible to use stochastic Navier-Stokes equations [97], but in this section we apply simpler Kraichan model, that was described in detail in Chapter 5 and in compressible version in Chapter 6. Let just note that $\mathbf{v}(x)$ is a random Gaussian variable

with zero mean and the correlator

$$\langle v_i(x)v_j(x')\rangle = \int \frac{d\mathbf{k}d\omega}{(2\pi)^{d+1}} P_{ij}(\mathbf{k})D_v(\omega,\mathbf{k})\exp[-i\omega(t-t')+i\mathbf{k}.(\mathbf{x}-\mathbf{x}')]. \tag{8.7}$$

with kernel function

$$D_v(\omega,\mathbf{k}) = \frac{g_{10}D_0^3 k^{4-d-2\epsilon-\eta}}{\omega^2+u_0^2D_0^2(k^{2-\eta})^2}. \tag{8.8}$$

The main difference with the annihilation problem is that now critical dimension is $d_c = 4$ instead of $d_c = 2$. In what follow we will employ dimensional regularization, therefore in the (8.8) one should substitute for the space dimension d expression $d = 4-2\Delta$ (the factor 2 was chosen only for technical convenience). Exponent Δ will play role of expansion parameter (similarly as ϵ and η) and will appear as a pole in the perturbative expansion of renormalization constants. The averaging procedure with respect to the velocity field $\mathbf{v}(x)$ with correlator (8.8) may be performed with the aid of the following action functional

$$S_{20} = -\frac{1}{2}\int dt\, dt'\, d\mathbf{x}\, d\mathbf{x}'\, \mathbf{v}(t,\mathbf{x})D_v^{-1}(t-t',\mathbf{x}-\mathbf{x}')\mathbf{v}(t',\mathbf{x}'), \tag{8.9}$$

where D_v^{-1} is the inverse correlator (8.7) (in the sense of the Fourier transform). The expectation value of any relevant physical observable may be calculated using the complete weight functional $\mathscr{W}(\psi^\dagger,\psi,\mathbf{v}) = \mathrm{e}^{S_{10}+S_{20}}$, where S_{10} and S_{20} are the action functionals (8.3) and (8.9).

The full problem is equivalent to the field-theoretic model of three fields $\Phi = \{\psi^\dagger,\psi,\mathbf{v}\}$ with the total action functional given as the sum of (8.3) and (8.9)

$$S_0(\Phi) = S_{10}(\Phi) + S_{20}(\Phi). \tag{8.10}$$

Formulation (8.10) together with (8.9) and (8.3) gives a possibility to calculate statistical averages of random quantities as functional averages with the weight $\exp S_0(\Phi)$. The generating functionals of total $G(A)$ and connected $W(A)$ Green functions can be represented as functional (path) integral

$$G(A) = \exp W(A) = \int \mathscr{D}\Phi \exp[S_0(\Phi) + A\Phi] \tag{8.11}$$

with sources $A = \{A_v, A_{\psi^\dagger}, A_\psi\}$ as a scalar product $A\Phi = \sum_\Phi A_\Phi\Phi$. All correlation and response functions could be calculated in perturbation fashion by a standard procedure

[1, 2]. The general consequence of causality is the vanishing of Green functions of the following form

$$\langle \psi^\dagger(t_1,\mathbf{x}_1)\psi^\dagger(t_2,\mathbf{x}_2)\dots\psi^\dagger(t_N,\mathbf{x}_N)\rangle, \tag{8.12}$$

which should be satisfied for arbitrary N and any stochastic model [1]. Let us note that the inclusion of the velocity field does not break the symmetry (8.5). Hence vanishing of Green functions $\langle \psi(t_1,\mathbf{x}_1)\psi(t_2,\mathbf{x}_2)\dots\psi(t_N,\mathbf{x}_N)\rangle$ can be viewed as a consequence of the symmetry (8.5). This symmetry is also responsible for λ_0^2 being the real expansion parameter rather than single λ_0 as can be seen by the direct inspection of Feynman graphs. Therefore it is advisable to introduce new charge as

$$g_{20} = \lambda_0^2 . \tag{8.13}$$

In the literature[81, 95] it corresponds to the charge u_0.

8.3 SCALING ANALYSIS AND UV RENORMALIZATION PROCEDURE

The theoretical analysis of the UV divergences is based on the power counting analysis [2]. The main difference with previous sections lies in the introduction of new parameters τ_0 and λ_0, or g_{20} respectively. However the method for calculation of canonical dimensions is the same as already used in Sections 4-6. The canonical dimensions of

Q	ψ	$\psi^\dagger$	v	g_{10}	λ_0	g_{20}	D_0	u_0	τ_0
d_Q^k	$d/2$	$d/2$	-1	$2\epsilon+\eta$	Δ	2Δ	-2	η	2
d_Q^ω	0	0	1	0	0	0	1	0	0
d_Q	$d/2$	$d/2$	1	$2\epsilon+\eta$	Δ	2Δ	0	η	2

Table 6: Canonical dimensions of the fields and bare parameters

all parameters and fields for the model (8.10) are given in the Table 6. We see that the model is logarithmic (the canonical dimensions of the coupling constants g_{10}, g_{20} and u_0 simultaneously vanish) at space dimension $d = 4$ (or equivalently $\Delta = 0$) and for the choice $\epsilon = \eta = 0$. In what follows we will employ dimensional regularization with the minimal subtraction (MS) scheme. According to the general theory [2] of the renormalization group the UV divergences in the Green functions in this scheme manifest themselves as

poles in ϵ, δ, η or possibly as their linear combination.
The total canonical dimension for arbitrary one-particle irreducible (1PI) Green function $\Gamma = \langle \Phi \dots \Phi \rangle_{1-ir}$ is given by the relation (2.36)

$$d_\Gamma = d + 2 - N_\Phi d_\Phi \,, \tag{8.14}$$

where $N_\Phi = \{N_v, N_{\psi^\dagger}, N_\psi\}$ are the numbers of the external fields entering into the Green function Γ and the summation over all types of fields is implied. From the symmetry (8.5) it follows that the counterms corresponding to the terms $(\psi^\dagger)^2\psi$ and $\psi^\dagger\psi^2$ have to be renormalized by the same renormalization constant. We are thus led to conclusion that all terms that should be renormalized are already present in the action. Therefore we conclude that the model (8.10) is multiplicatively renormalizable and the renormalized action can be written as

$$S_R(\Phi) \;=\; \psi^\dagger[-Z_1\partial_t - Z_1(\mathbf{v}.\boldsymbol{\nabla}) + Z_2 D\nabla^2 - Z_3 D\tau]\psi + \frac{Z_4 D\lambda}{2}[(\psi^\dagger)^2\psi - \psi^\dagger\psi^2] + \frac{1}{2}\mathbf{v}D_v^{-1}\mathbf{v}. \tag{8.15}$$

The renormalization action is obtained from the bare one by the multiplicative renormalization of the fields $\psi^\dagger \to Z_{\psi^\dagger}\psi^\dagger, \psi \to Z_\psi\psi$ (it turns out that Z_v=1) and the parameters

$$D_0 = DZ_D \,, \quad g_{10} = g_1\mu^{2\epsilon+\eta}Z_{g_1} \,, \quad \tau_0 = \tau Z_\tau \,, \quad u_0 = u\mu^\eta Z_u \,, \quad g_{20} = g_2\mu^{2\Delta}Z_{g_2} \,. \tag{8.16}$$

The renormalization constant for the parameter λ_0 can be easily derived using definition (8.13) to obtain $Z_{g_2} = Z_\lambda^2$. From the renormalized action (8.15) and the definition (8.16) it is easy to relate renormalization constants as follows

$$\begin{aligned} &Z_1 = Z_\psi Z_{\psi^\dagger} = Z_\psi Z_{\psi^\dagger} Z_v \,, \quad Z_2 = Z_\psi Z_{\psi^\dagger} Z_D \,, \quad Z_3 = Z_\psi Z_{\psi^\dagger} Z_\tau Z_D \,, \\ &Z_4 = Z_D Z_\lambda Z_{\psi^\dagger}^2 Z_\psi = Z_D Z_\lambda Z_{\psi^\dagger} Z_\psi^2 \,, \quad Z_5 = Z_v \,. \end{aligned} \tag{8.17}$$

Since the nonlocal term involving $\mathbf{v}$ fields in (8.15) should not be renormalized, the relations $Z_{g_1}Z_D^3 = 1$ and $Z_u Z_D = 1$ have to be satisfied. Inverting relations (8.17) leads to the relations for original parameters of the model

$$\begin{aligned} &Z_v = 1, \quad Z_\psi = Z_{\psi^\dagger} = Z_1^{1/2}, \quad Z_\lambda = Z_4 Z_2^{-1} Z_1^{-1/2}, \quad Z_D = Z_2 Z_1^{-1}, \\ &Z_u = Z_1 Z_2^{-1}, \quad Z_\tau = Z_3 Z_2^{-1}, \quad Z_{g_1} = Z_1^3 Z_2^{-3}, \quad Z_{g_2} = Z_4^2 Z_1^{-1} Z_2^{-2}. \end{aligned} \tag{8.18}$$

8.4 CALCULATION OF THE RENORMALIZATION CONSTANTS

The standard perturbative approach is based on the diagrammatic expansion into the Feynman graphs [1, 2]. The inverse matrix of the free (quadratic) part of the actions (8.3) and (8.9) determines the form of the bare(unrenormalized) propagators. It is easy

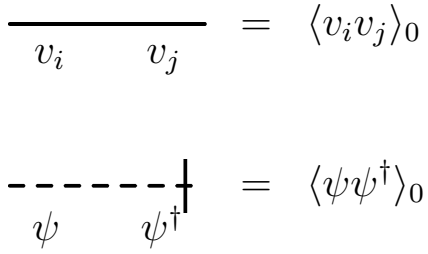


Figure 15: The propagators of the bare model calculated from the quadratic part of actions (8.3) and (8.9)

to see that the studied model contains three different types of propagators graphically depicted in Figure 15. In the frequency-momentum representation they are given by the expressions

$$\langle v_i v_j \rangle_0 = \frac{g_{10} D_0^3 k^{4-d-2\epsilon-\eta}}{\omega^2 + [u_0 D_0 k^{2-\eta}]^2}, \tag{8.19}$$

$$\langle \psi \psi^\dagger \rangle_0 = \langle \psi^\dagger \psi \rangle_0^* = \frac{1}{-i\omega + D_0(k^2 + \tau_0)}. \tag{8.20}$$

This has to be contrasted with the models [89, 94] where the inclusion of long-range interactions led just to the redefining $\psi^\dagger \psi$ propagator. The propagators for the model (8.10) have the form (5.7) and the vertex factor [1]

$$V_m(x_1, x_2, \ldots, x_m; \Phi) = \frac{\delta^m V(\Phi)}{\delta\Phi(x_1)\delta\Phi(x_2)\ldots\delta\Phi(x_m)} \tag{8.21}$$

is associated to each interaction vertex of Feynman graph. Here, Φ could be any member from the full set of fields $\{\psi^\dagger, \psi, v\}$. From the action (8.3) three possible interaction terms are produced and are graphically depicted in Figure 16. First two of them are the usual interaction vertices for directed percolation problem. The last one is responsible for the advection of spreading agent by the velocity field and we have introduced it already in equation (4.10). In the momentum representation it is given by the expression (4.11). From the explicit form of propagators and interaction vertices perturbation series in Feynamn diagrams could be constructed for the 1PI Green functions. UV finitness

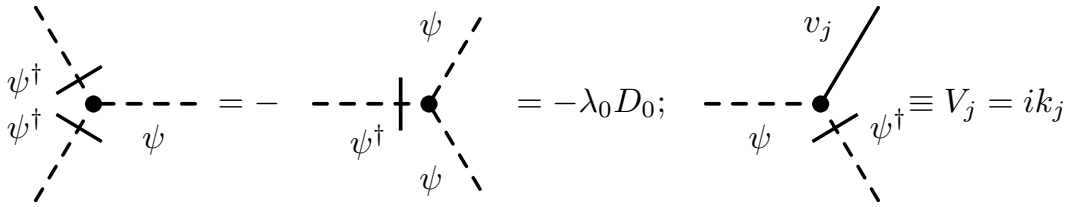


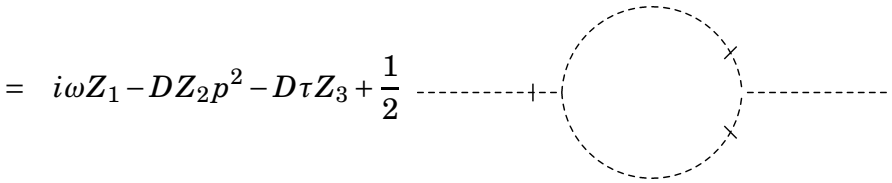
Figure 16: Interaction vertices obtained from the nonlinear terms from bare actions (8.3) and (8.9)

condition for 1PI functions $\Gamma_{\psi^\dagger\psi}$ and $\Gamma_{(\psi^\dagger)^2\psi}$ leads to the determination of all renormalization constants $Z_1 - Z_4$. The Dyson equation [1] for the 1PI function $\Gamma_{\psi^\dagger\psi}$ is given by the expression

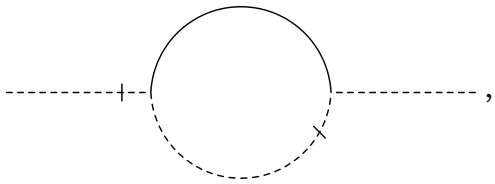
$$\Gamma_{\psi^\dagger\psi} = -\Delta^{-1}_{\psi\psi^\dagger} + \Sigma_{\psi\psi^\dagger}\,, \tag{8.22}$$

where $\Sigma_{\psi\psi^\dagger}$ denotes self-energy. In one-loop approximation Dyson equation can be graphically written as

$$\Gamma_{\psi^\dagger\psi} = i\omega Z_1 - DZ_2 p^2 - D\tau Z_3 + \frac{1}{2}$$



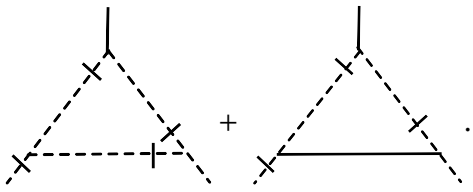
$$+$$



$$, \tag{8.23}$$

where the prefactor in the first graphs represents its symmetry coefficient. Owing to the Galilean symmetry, the constant Z_1 can also be determined from the renormalization of advecting triple term $\psi^\dagger\psi v$

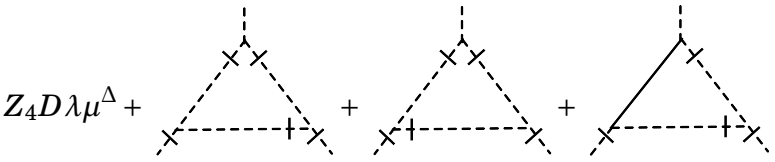
$$\Gamma_{\psi^\dagger\psi v} = ip_1 Z_1 +$$

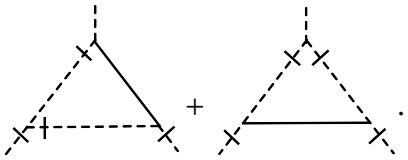


$$+$$

$$. \tag{8.24}$$

Finally the perturbation expansion for the 1PI function $\Gamma_{(\psi^\dagger)^2\psi}$ can be written as

$\Gamma_{(\psi^\dagger)^2\psi} = Z_4 D\lambda\mu^\Delta +$  (8.25)

 (8.26)

The first two graphs differ from each other only in the direction of one internal propagator, what implies their numerical equality. The next three graphs are UV convergent, which is a result of tranversal character of velocity propagator vv. Consequently only the first two graphs gives the nontrivial contribution to the given renormalization constant Z_4. The explicit results for the ultraviolet divergent parts of Feynman graphs can be found in Appendix D. One-loop approximation leads in MS scheme to the following expressions

$$Z_1 = 1 + \frac{\overline{g_2}}{16\Delta}\,, \quad Z_2 = 1 + \frac{\overline{g_2}}{32\Delta} - \frac{3\overline{g_1}}{16u(1+u)\epsilon}\,, \quad Z_3 = 1 + \frac{\overline{g_2}}{8\Delta}\,, \quad Z_4 = 1 + \frac{\overline{g_2}}{4\Delta}\,, \tag{8.27}$$

where $\overline{g_i} = g_i\overline{S_4}(i = 1,2)$ with $\overline{S_d} = S_d/(2\pi)^d$ (S_d is the volume of unit sphere in d-dimension) is the common factor resulting from the momentum integration in dimensional regularization. It is useful to rescale the charges according to the prescription

$$\overline{g_1} \to 2g_1\,, \quad \overline{g_2} \to 2g_2\,, \tag{8.28}$$

which changes expressions (8.27) into the slightly different form

$$Z_1 = 1 + \frac{g_2}{8\Delta}\,, \quad Z_2 = 1 + \frac{g_2}{16\Delta} - \frac{3g_1}{8u(1+u)\epsilon}\,, \quad Z_3 = 1 + \frac{g_2}{4\Delta}\,, \quad Z_4 = 1 + \frac{g_2}{2\Delta}\,. \tag{8.29}$$

Using relations (8.18) the renormalization constants for the fields and parameters of the original model are easily derived (to the given precision order)

$$\begin{aligned}
&Z_\tau = 1 + \frac{3g_2}{16\Delta} + \frac{3g_1}{8u(1+u)\epsilon}\,, \quad Z_\psi = Z_{\psi^\dagger} = 1 + \frac{g_2}{16\Delta}\,,\\
&Z_D = 1 - \frac{3g_1}{8u(1+u)\epsilon} - \frac{g_2}{16\Delta}\,, \quad Z_\lambda = 1 + \frac{3g_2}{8\Delta} + \frac{3g_1}{8u(1+u)\epsilon}\,,\\
&Z_{g_2} = 1 + \frac{3g_2}{4\Delta} + \frac{3g_1}{4u(1+u)\epsilon}\,, \quad Z_u = 1 + \frac{g_2}{16\Delta} + \frac{3g_1}{8u(1+u)\epsilon}\,.
\end{aligned} \tag{8.30}$$

8.5 IR STABLE FIXED POINTS

The coefficient functions of the RG operator

$$D_{\mathrm{RG}} = \mu\frac{\partial}{\partial\mu}\Big|_0 = \mu\frac{\partial}{\partial\mu} + \sum_{g_i}\beta_i\frac{\partial}{\partial g_i} - \gamma_D D\frac{\partial}{\partial D}\,, \tag{8.31}$$

where the bare parameters are denoted with the subscript "0", beta functions are defined for the charges $g_i = \{g_1, g_2, u\}$ as

$$\beta_i = \frac{\partial g_i}{\partial \ln\mu}\Big|_0, \tag{8.32}$$

whereas anomalous dimension $\gamma_F, F \in \{D, g_2\}$ are defined as

$$\gamma_F = \frac{\partial \ln Z_F}{\partial \ln\mu}\Big|_0\,, \tag{8.33}$$

From this definition and from relations (8.16) follows the explicit form for the beta functions

$$\beta_{g_1} = g_1(-2\epsilon - \eta + 3\gamma_D), \quad \beta_u = u(-\eta + \gamma_D), \quad \beta_{g_2} = g_2(-2\Delta - \gamma_{g_2}) \tag{8.34}$$

and for the relevant anomalous dimensions γ_D and γ_{g_2}

$$\gamma_D = \frac{3g_1}{4u(1+u)} + \frac{g_2}{8}\,, \quad \gamma_{g_2} = -\frac{3g_1}{2u(1+u)} - \frac{3g_2}{2}\,. \tag{8.35}$$

The scaling regimes are associated with the fixed points of the corresponding RG functions. The fixed points g^* are defined as such points $g^* = (g_1^*, u^*, g_2^*)$ for which all β_g functions simultaneously vanish

$$\beta_{g_1}(g_1^*, u^*, g_2^*) = \beta_u(g_1^*, u^*, g_2^*) = \beta_{g_2}(g_1^*, u^*, g_2^*) = 0. \tag{8.36}$$

The type of the fixed point is determined by the eigenvalues of the matrix $\Omega = \{\Omega_{ik} = \partial\beta_i/\partial g_k\}$, where β_i is the full set of β functions (8.34) and g_k is the full set of charges $\{g_1, u, g_2\}$. The IR asymptotic behavior is governed by the IR stable fixed points, for which all eigenvalues of Ω matrix are positive.

It is easy to see that the functions β_{g_1} and β_u satisfy relation $\beta_{g_1}/g_1 - 3\beta_u/u = 2(\eta - \epsilon)$. This means that they cannot be equal zero simultaneously for the finite values of the charges g_1 and u. The only exception is the instance $\epsilon = \eta$, which should be studied separately. For general case $\epsilon \neq \eta$ we have to set either $u = 0$ or $u = \infty$ and rescale g in such a way,

that γ_D remains finite [21].

In what follows we present the results for fixed points, anomalous dimensions and eigenvalues of the Ω matrix to the first order of perturbation theory. However, we would like to stress, that the form of β functions (8.34) allows to calculate the anomalous dimensions γ_D and γ_λ exactly (without any second-order correction).

In [76] independence of the renormalization constant Z_D on the exponents η at the two-loop approximation has been conjectured. It implies that we may use the choice $\eta = 0$, which we have applied in our calculations of the renormalization constants Z_D and Z_{g_2}.

Let us consider the "rapid-change mode" ($u \to \infty$). It is convenient to introduce new charges $w = 1/u$, $g_1' = g_1/u^2$ and the corresponding β functions obtain the form

$$\beta_{g_1'} = g_1'(\eta - 2\epsilon + \gamma_D), \quad \beta_w = w(\eta - \gamma_D), \quad \beta_{g_2} = g_2(-2\Delta - \gamma_{g_2}). \tag{8.37}$$

The "rapid-change model" corresponds to the fixed point with $w^* = 0$. In this case four stable IR fixed points are realized:

$$\begin{aligned}
&\textbf{FP 1A:} && w^* = 0, \quad {g_1'}^* = 0, \quad g_2^* = 0, && (8.38)\\
& && \gamma_D = 0, \quad \gamma_{g_2} = 0, &\\
&\text{stable for} && \eta > 2\epsilon, \quad \Delta < 0, \quad \eta > 0. &\\
&\textbf{FP 1B:} && w^* = 0, \quad {g_1'}^* = 0, \quad g_2^* = \frac{4\Delta}{3}, && (8.39)\\
& && \gamma_D = \frac{\Delta}{6}, \quad \gamma_{g_2} = -2\Delta, &\\
&\text{stable for} && 6\eta + \Delta > 12\epsilon, \quad \Delta > 0, \quad \Delta > 6\eta. &\\
&\textbf{FP 2A:} && w^* = 0, \quad {g_1'}^* = \frac{4(2\epsilon - \eta)}{3}, \quad g_2^* = 0, && (8.40)\\
& && \gamma_D = 2\epsilon - \eta, \quad \gamma_{g_2} = 2\eta - 4\epsilon, &\\
&\text{stable for} && 2\epsilon > \eta > \epsilon, \quad 2\epsilon > \eta + \Delta. &\\
&\textbf{FP 2B:} && w^* = 0, \quad {g_1'}^* = \frac{4(12\epsilon - 6\eta - \Delta)}{15}, \quad g_2^* = \frac{8(\Delta + \eta - 2\epsilon)}{5}, && (8.41)\\
& && \gamma_D = 2\epsilon - \eta, \quad \gamma_{g_2} = -2\Delta, &\\
&\text{stable for} && \Delta > 0, \quad \Delta + \eta > 2\epsilon, \quad 12\epsilon > 6\eta + \Delta. &
\end{aligned}$$

For the analysis of the regime $u \to 0$ (quenched velocity field) we introduce the new charge ${g_1}'' \equiv g_1/u$. Hence the corresponding β functions have the form

$$\beta_{{g_1}''} = {g_1}''(-2\epsilon + 2\gamma_D),, \quad \beta_u = u(-\eta + \gamma_D), \quad \beta_{g_2} = g_2(-2\Delta - \gamma_{g_2}). \tag{8.42}$$

Also in this case there are four possible IR stable fixed points:

$$\textbf{FP 3A:} \quad u^* = 0, \quad g_1''^* = 0, \quad g_2^* = 0, \tag{8.43}$$
$$\gamma_D = 0, \quad \gamma_{g_2} = 0,$$
$$\text{stable for} \quad \epsilon < 0, \quad \Delta < 0, \quad \eta < 0.$$
$$\textbf{FP 3B:} \quad u^* = 0, \quad g_1''^* = 0, \quad g_2^* = \frac{4\Delta}{3}, \tag{8.44}$$
$$\gamma_D = \frac{\Delta}{6}, \quad \gamma_{g_2} = -2\Delta,$$
$$\text{stable for} \quad \epsilon < 0, \quad \Delta > 0, \quad \eta < 0.$$
$$\textbf{FP 4A:} \quad w^* = 0, \quad g_1''^* = \frac{4\epsilon}{3}, \quad g_2^* = 0, \tag{8.45}$$
$$\gamma_D = \epsilon, \quad \gamma_{g_2} = -2\epsilon,$$
$$\text{stable for} \quad \epsilon > 0, \quad \epsilon > \Delta, \quad \epsilon > \eta.$$
$$\textbf{FP 4B:} \quad w^* = 0, \quad g_1''^* = \frac{4(6\epsilon - \Delta)}{15}, \quad g_2^* = \frac{8(\Delta - \epsilon)}{5}, \tag{8.46}$$
$$\gamma_D = \epsilon, \quad \gamma_{g_2} = -2\Delta,$$
$$\text{stable for} \quad 0 < \epsilon < \Delta < 6\epsilon, \quad \epsilon > \eta.$$

In the special case $\epsilon = \eta$ the functions β_{g_1} and β_u become proportional and this leads to the degeneration of fixed point. Instead of just plain fixed point, we observe a whole line of fixed points in the (g_1, u) plane.

$$\textbf{FP 5A:} \quad \frac{g_1^*}{u^*(1+u^*)} = \frac{4\epsilon}{3}, \quad g_2^* = 0 \tag{8.47}$$
$$\gamma_D = \epsilon = \eta, \quad \gamma_{g_2} = -2\epsilon$$
$$\text{stable for} \quad \eta = \epsilon > \Delta, \quad \epsilon > 0$$
$$\textbf{FP 5B:} \quad \frac{g_1^*}{u^*(1+u^*)} = \frac{4(6\epsilon - \Delta)}{15}, \quad g_2 = \frac{8(\Delta - \epsilon)}{5} \tag{8.48}$$
$$\gamma_D = \epsilon, \quad \gamma_{g_2} = -2\Delta$$
$$\text{stable for} \quad \epsilon = \eta < \Delta, \quad \epsilon > 0.$$

The “real problem” corresponding to the Kolmogorov scaling is obtained for the value $\eta = \epsilon = 4/3$, which leads to the famous “five-thirds law” [15] for the spatial velocity statistics. By direct observation we see that in this case critical behavior is described by the fixed point 5A for logarithmic ($\Delta = 0, d = 4$), three dimensional ($\Delta = 1/2, d = 3$) and also two dimensional ($\Delta = 1, d = 2$) case. Fixed point **5A** is characterised by vanishing of the charge g_2. Therefore we conclude that in the turbulent field cubic interactions in the

vicinity of critical point (8.3) are negligible.

8.6 CONCLUSIONS

This section was devoted to the study of directed percolation problem influenced by the external advecting velocity field. In order to use the technique of the perturbative renormalization a field-theoretic model is constructed. All the calculations were performed to the first order of the perturbation theory. The IR stable fixed points, dimensions and corresponding regions of stability of fixed points are calculated. The technically relatively simple model of velocity fluctuations used here is a convenient starting point for more realistic high-loop calculations.

9 DISCUSSION

In this thesis we have presented several results concerning annihilation process in the presence of advecting field of different origin. We also made contribution to the problem of directed bond percolation. The main theoretical tool was functional integral representation, Feynman diagrammatic expansion, UV renormalization technique and dimensional regularization in the minimal subtraction scheme for the calculation of renormalization constants. After renormalization of the model by the means of RG approach, fixed points' structure was obtained and the corresponding regions of stability for each of fixed points were calculated. The main contributions are summarized below.

- In the presence of velocity fluctuations generated by stochastic Navier-Stokes equations annihilation process $A + A \to \varnothing$ was analyzed using field-theoretic approach. After succesful construction of effective action functional, perturbative renormalization group was applied for analysing of asymptotic behavior. Model was succesfully renormalized to the second order approximation, fixed points' structure, regions of stability was also obtained to the given order. Nontrivial corrections to the rate equation coming from the higher order perturbation terms were estimated. Integro-differential equation for the mean particle number was derived.

- Effect of random fluctuations described by Kraichnan ensemble on the annihilation process was studied with the help of UV renormalization of the model. Renormalization together with coordinates of fixed points and their regions of stability were calculated to the second order precision.

- Compressibility of environment was taken into account and it was shown that in some regimes it leads to the enhancement of the annihilation process.

- General model for processes with non-conserved number of particles was analysed in the presence of sinks and sources.

- Directed bond percolation problem was considered in the presence of velocity fluctuations created by Kraichnan model with finite correlation time.

There remains some open questions, which we hope to solve in the future:

- Construction of field theoretic model for bimolecular reaction process $A + B \to \varnothing$ and its renormalization

- Higher order loop calculation for the annihilation process with compressible Kraichnan model to acknowledge one-loop results and confirm stability of the large-scale regimes
- Higher order loop calculation for the percolation problem with finite correlated Kraichnan model to acknowledge one-loop results and confirm stability of the large-scale regimes

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10 RESUMÉ

Na rozdiel od rovnovážnej štatistickej fyziky existuje len málo exaktných výsledkov pre nerovnovážne systémy. Očakávame ale, že tak ako pre rovnovážne aj pre nerovnovážne systémy môžu existovať jednoduchšie situácie (fázové prechody druhého druhu), kedy ich je možné charakterizovať malým počtom univerzálnych parametrov, ku ktorým patria priestorová dimenzia, symetria a počet komponentov parametra usporiadania. Neznalosť funkcionálu voľnej energie ale spôsobuje, že na analýzu možného správania je nutné použiť sofitikovanejšie metódy ako pre rovnovážne systémy. Ako jeden z najefektívnejších postupov sa javí konštrukcia kvantovo-poľového modelu [1, 2], vychádzajúc či už zo stochastickej Langevinovej rovnice alebo riadiacej rovnice. Na získaný poľový účinok je následne možné aplikovať metódu renormalizačnej grupy, ktorá bola veľmi úspešná pre popis kritického správania sa rovnovážnych systémov.

Nerovnovážne systémy je možné vo všeobecnosti rozdeliť [11] do dvoch tried:

(a) systémy s hermitovským hamiltoniánom, ktorých stacionárne stavy sú popísané pomocou Gibbsovej-Boltzmannovej distribučnej funkcie, ktoré sa nachádzajú v stave ďaleko od termodynamickej rovnováhy. Pre takéto systémy je možné získať dynamický popis priamočiarym zovšeobecnením statických modelov. Ako príklady môžeme uviesť Landauovu-Ginzburgovu rovnicu pre časový priebeh lokálnej magnetizácie, kinetický Isingov model, modely typu A až J kritickej dynamiky [12].

(b) systémy s nehermitovským hamiltoniánom, resp. bez hamiltoniánu vôbec, pričom pre tieto systémy vo všeobecnosti nemusí existovať termodynamický rovnovážny stav. Podmienka detailnej rovnováhy pre ne nie je splnená a teda ani Einsteinove vzťahy medzi tepelnými fluktuáciami a trecími silami nemôžu platiť. Medzi typické príklady patria tekutiny v turbulentom stave, nevratné chemické procesy, rôzne modely rastu povrchov, turbulentná advekcia atď. Pre túto triedu modelov je nutné aplikovať iné postupy ako napríklad zovšeobecnenú Langevinovu rovnicu alebo riadiacu rovnicu [13]. Pre prvý z nich je nutné vhodným spôsobom na základe fenomenologických pozorovaní zvoliť vlastnosti náhodnej sily, ktorá simuluje rozhodujúce vlastnosti exaktného mikroskopického modelu. Druhý prístup je pravdepodobne viac rigorózny, ale zároveň matematicky ťažšie zvládnuteľný.

Podrobnejšie sa zamerajme na problémy triedy b), ktorých štúdium tvorí hlavnú náplň predkladanej dizertačnej práce. Porozumenie prúdeniu tekutín vo všeobecnom prípade je jeden z posledných problémov klasickej fyziky, ktorého riešenie nepoznáme. Tekutiny

môžu vykazovať veľmi odlišné správanie sa [15], od veľmi jednoduchého ako laminárne prúdenie po veľmi komplikované realizované napríklad vo vzdušných víroch. Nestabilita Navierových-Stokesových rovníc [14] vzhľadom na počiatočné podmienky spôsobuje nemožnosť spoľahlivej predikcie jej riešení, s ktorou sme veľmi dobre oboznámení v súvislosti s nemožnosťou predpovede počasia na viac ako pár dní vopred. Pre klasifikáciu režimov, v ktorých sa tekutina môže vyskytovať, bolo zavedené bezrozmerné Reynoldsovo číslo Re. Je definované ako podiel $\mathrm{Re} = VL/\nu$, kde V je typická stredná rýchlosť prúdenia, L reprezentatívna škála v danom probléme (napr. rozmer prekážky, ktorá spôsobuje perturbácie) a ν je kinematická viskozita tekutiny. Ide teda o pomer medzi zotrvačnými a trecími (disipačnými) silami.

Pre malé hodnoty $\mathrm{Re} \ll 1$ nastáva regulárne (laminárne) prúdenie. So zvyšujúcou sa hodnotou Re sú pozorované veľmi odlišné javy od periodických pohybov, ako sú Kármánove víry, až po nepravidelné chaotické pohyby v limite veľkych hodnôt $\mathrm{Re} \gg 1$ (v praxi sú za dostatočne veľké považované hodnoty $\mathrm{Re} \geq 10^6$) [15, 16]. Tento stav tekutiny je známy ako plne rozvinutá turbulencia. V takomto režime je vplyv počiatočných podmienok zanedbateľný a štatistické vlastnosti charakteristík tekutiny (ako sú korelačné a štruktúrne funkcie rýchlostného poľa) vykazujú univerzálne správanie. Na prvý pohľad beznádejne komplikovaný problém sa ukáže ako teoreticky ľahšie popísateľný vďaka dynamickej emergencii nových symetrií, ktoré sú štatistickej povahy. V prácach [17, 18] Kolmogorov postuloval hypotézy, pomocou ktorých je možné vysvetliť mnohé javy týkajúce sa turbulentného správania sa a taktiež predikovať rôzne škálovacie zákonitosti.

V istom zmysle Kolmogorovova teória predstavuje teóriu ideálnej turbulencie, pretože je formulovaná pre prípad nekonečnej hodnoty Reynoldsovho čísla $\mathrm{Re} \to \infty$. Táto teória nebola do dnešných dní rigorózne dokázaná z prvých princípov, v tomto prípade z Navierovej-Stokesovej rovnice.

Stochastická Navierova-Stokesova (ďalej SNS) rovnica bola navrhnutá za účelom potvrdenia Kolmogorovských hypotéz. Hlavný rozdiel oproti modelom kritickej dynamiky spočíva v pochopení toho, že pre turbulenciu nie je známy funkcionál voľnej energie, čo vyplýva z neexistencie jej statického (termodynamicky rovnovážneho) ekvivalentu. SNS rovnica zanedbáva javy ako je priamy vplyv okrajových podmienok, ktoré sú zodpovedné za vytvorenie turbulentného stavu. Tieto sú modelované pôsobením náhodnej sily. Táto sila je v SNS rovnici taktiež dôležitá pre matematický popis sústavnej dodávky energie, bez ktorej by nebolo možné mať tekutinu v stacionárnom stave. Vďaka prítomnosti viskozity vždy dochádza k disipácii energie. Tieto efekty sú fenomenologicky popísané vhodným zadaním štatistických vlastností náhodnej sily. Napriek tomu, že dô-

kaz ekvivalencie medzi SNS rovnicou a Kolmogorovskými hypotézami stále chýba, je možné pomocou SNS rovnice predpovedať mnohé škálovacie vlastnosti korelačných funkcií rýchlostného poľa.

Ďalší zaujímavý problém v súvislosti s turbulenciou je spojený s advekciou (prenosom) nejakej veličiny [19]-[21] (teplotné pole, pole koncentrácie, znečistenie v atmosfére a pod.) spôsobenej rýchlostným poľom. Tento problém je okrem nespochybniteľnej praktickej dôležitosti zaujímavý aj z teoretického hľadiska. Intermitentná (fraktálna) povaha korelačných funkcií sa oveľa výraznejšie prejavuje v tzv. Kraichnanovom modeli pasívnej prímesi ako i v značne komplikovanejšom modeli popísanom SNS rovnicou.

Procesy s nezachovavajúcim sa počtom častíc, ako napríklad nevratné chemické reakcie [13], predstavujú iné možné realizácie triedy b). Je potrebné zdôrazniť, že nevratnosť je dôležitým atribútom týchto systémov. Všeobecná vratná chemická reakcia je veľmi dobre popísateľná pomocou kinetickej rovnice. Z fundamentálneho hľadiska je možné túto zákonitosť chápať ako dôsledok splnenia podmienky ergodicity, nakoľko všetky možné stavy (s rôznym počtom reagujúcich častíc) môžu byť v každom časovom okamihu dosiahnuté. V dlhočasovej asymptotike $t \to \infty$ (ktorá nás vždy zaujíma z praktického hľadiska) koncentrácie reaktantov a produktov potom zjavne nadobúdajú svoje rovnovážne hodnoty. Avšak pre nevratné chemické reakcie sa fázový priestor (počet možných stavov) neustále zmenšuje. To je príčinou nemožnosti použitia zvyčajných metód rovnovážnej štatistickej fyziky.

Jeden z možných postupov je založený na použití Langevinovej rovnice [13] pre pole koncentrácie. Niektoré práce ale ukazujú neadekvátnosť tohto prístupu. Napríklad pre anihilačný proces $2A \to \varnothing$ predpovedá hodnotu kritickej dimenzie $d_c = 6$ [22], čo je v príkrom rozpore s hodnotou $d_c = 2$, ktorá je potvrdená škálovacou analýzou [23] a tiež metódou renormalizačnej grupy [24].

Iný problém s týmito procesmi je spôsobený nízkou hodnotou ich kritických dimenzií (máme na mysli jednorozmerný $d = 1$, resp. dvojrozmerný $d = 2$ priestor). Ak je pohyb častíc zapríčinený difúziou (ktorá je vďaka teplotným fluktuáciám prítomná takmer vo všetkých fyzikálnych podmienkach), vlastnosť neustálej návratnosti [25] difúzie spôsobuje značné fluktuácie (antikorelácie) v koncentrácii reagujúcich častíc. Známy Doiov prístup [26] sa ukazuje ako veľmi efektívny nástroj na popis klasických systémov s premenlivým počtom častíc. Je založený na vhodnej projekcii klasického systému na kvantový systém. Po príslušnom skonštruovaní Fockovho priestoru kvantových stavov, je možné pomocou koherentných stavov nájsť reprezentáciu vo forme dráhového intregrálu s nejakým efektívnym účinkom. Na jeho analýzu je následne možné použiť metódu renor-

malizačnej grupy a tým získať informáciu o asymptotických vlastnostiach riešení príslušného modelu. Chemické reakcie zvyčajne prebiehajú v nejakom vonkajšom prostredí, napr. v kvapalnom roztoku, polutanty v ovzduší, spaľovacie procesy v motoroch atď. A toto prostredie sa môže samozrejme nachádzať v rôznych stavoch. Vyvstávajú preto prirodzené otázky: aký je vplyv prostredia na priebeh takýchto procesov? Ma prostredie vždy urýchľujúci účinok, ako by sme naivne očakávali? Aké parametre naozaj kontrolujú priebeh reakcií? Prostredie môže byť zodpovedné za vznik napríklad tepelných fluktuácii, atmosferických prúdov alebo náhodných porúch. Zadaním štatistických vlastností prostredia základnou úlohou je kvantifikovať jeho vplyv na časový priebeh chemických procesov, ktoré sa v ňom odohrávajú. Z praktických aj teoretických dôvodov sa zvyčajne študujú veľkoškálové asymptotické vlastnosti nerovnovážnych systémov. Pod veľkými škálami máme na mysli veľké hodnoty pre priestorovú aj časovú zložku blízke typickým vonkajším mierkam.

K jedným z najefektívnejších prístupov pri štúdiu nerovnovážnych systémov patrí metóda renormalizačnej grupy. Tá umožňuje určiť stabilné veľkoškálové režimy a vypočítať kritické indexy vo forme asymptotických radov. Taktiež je tu neustála snaha o dosiahnutie čo najpresnejšej predikcie pre hodnoty anomálnych korekcií. Hlavné dôvody tejto snahy sú nasledovné:

- Prirodzená potreba určiť štruktúru asymptotických radov použitím vypočítaných koeficientov v danom priblížení, prípadne spočítať daný rad aspoň čiastočne.
- Vytvorenie možnosti pre porovnanie výsledkov s exaktne známymi a tým získanie nezávislého potvrdenia výsledkov.
- Vytvorenie možnosti pre porovnanie výsledkov poruchovej teórie v rámci rôznych schém a tak dospieť k efektívnejším výpočtovým postupom.
- Možnosť porovnania získaných výsledkov s numerickými a experimentálnymi údajmi a týmto spôsobom dospieť k lepšiemu pochopeniu fyzikálnej podstaty študovaného problému.

Hlavnou úlohou tejto dizertačnej práce bude štúdium vybranej triedy reakčno-difúznych procesov v prítomnosti advekčného rýchlostného poľa rôznej podstaty. Po skonštruovaní efektívnych teoreticko poľových modelov bude na ne aplikovaná metóda renormalizačnej grupy. Ciele dizertačnej práce sú nasledovné:

(a) Analýza vplyvu turbulentých fluktuácií generovaných stochastickou Navierovou-Stokesovou rovnicou na priebeh anihilačnej reakcie dvoch rovnakých molekúl $A+A \to \varnothing$. Výpočet renormalizačných konštánt, funkcií, pevných bodov a efektívneho Prandtlovho čísla v dvojslučkovom priblížení.

(b) Popis asymptotických vlastností poľa koncentrácie v časovej limite $t \to \infty$ s určením jednoslučkovej korekcie ku kinetickej rovnici v priblížení stredného poľa.

(c) Štúdium vplyvu rýchlostných fluktuácii popísaných pomocou Kraichnanovho modelu s konečným korelačným časom na časový vývoj anihilačnej reakcie $A + A \to \varnothing$ a analýza vplyvu stlačitelnosti prostredia na veľkoškálové správanie sa systému.

(d) Štúidum vplyvu náhodných zdrojov a únikových mechanizmov v riadiacej rovnici na priebeh chemických reakcií.

(e) Štúdium vplyvu náhodných fluktuácii na stabilitu fažového prechodu v perkolačnom probléme typu orientovaná väzba. Konštrukcia teoreticko-poľového modelu a výpočet renormalizačných veličín s presnosťou do prvého rádu poruchovej teórie.

Postup, ktorý sme zvolili pri riešení spomenutých problémov je založený na možnosti prechodu od zadanej klasickej úlohy k efektívnemu teoreticko-poľovému modelu. Ten umožňuje použitie metódy renormalizačnej grupy na analýzu jeho asymptotických vlastností v infračervenej oblasti (ďalej IR), to znamená oblasť malých frekvencií a hybností alebo ekvivalentne oblasť veľkých časových a priestorových škál (mierok). Pri výpočte efektívnej difúzie, reakčnej konštanty a škálovacích režimov bola použitá metóda ultrafialovej renormalizácie, rozmerovej a analytickej regularizácie v tzv. schéme minimálneho odčítania (MS)[2]. Vzhľadom na to, že nestlačiteľné rýchlostné pole nemá vplyv na renormalizáciu reakčnej konštanty v jednoslučkovom priblížení, bolo potrebné prejsť k výpočtom v dvojslučkovom priblížení. Na vyšetrenie správania sa stabilných režimov v IR oblasti a príslušných kritických indexov pre prípad stlačitelného rýchlostného poľa boli použité tie isté metódy ako pre prípad nestlačiteľného poľa. Ako postačujúce sa ukázalo už jednoslučkové priblíženie pre určenie netriviáleno vplyvu rýchlostného poľa na kritické správanie sa systému.

Cieľom tejto práce bolo študovať vplyv náhodného rýchlostného poľa na priebeh dvoch typov reakčno-difúznych procesov. Dosiahnuté pôvodné výsledky je možné zhrnúť do nasledujúcich bodov:

- Vykonali sme analýzu vplyvu fluktuácii hustoty častíc a rýchlostí na triedu univerzality kinetického procesu $A + A \rightarrow \emptyset$ použitím kvantovo-poľovej renormalizačnej grupy. Vypočítali sme škálovaciu funkciu a rozpadový exponent α (definovaný asymptotickým vzťahom $n(t) \overset{t\rightarrow\infty}{\propto} t^{-\alpha}$) pre stredný počet častíc pre štyri asymptotické režimy predpovedané renormalizačnou grupou:

  (i) triviálny (Gaussovský) režim s $\alpha_G = 1$,

  (ii) teplotný (krátko dosahový) režim s $\alpha_T = 1 + \Delta/2 + \Delta^2/2$,

  (iii) kinetický režim I s $\alpha_I = 1 + (3\Delta + \epsilon)/(3 - \epsilon)$,

  (iv) kinetický režim II s $\alpha_{II} = 1$.

  Parameter Δ označuje odklon od priestorovej dimenzie $d = 2$ podľa vzťahu $d = 2+2\Delta$ a parameter ϵ určuje mieru vplyvu veľkoškálových módov rýchlostného poľa na reakčné procesy. Vypočítali sme renormalizačné konštanty do druhého rádu poruchovej teórie a našli rozpadový koeficient pre koncentráciu častíc do daného rádu v rozvoji po ϵ, Δ pre štyri IR (oblasť malých vlnových vektorov) stabilné pevné body RG, ktorých oblastí (znázornené na obr.10 spolu s naznačenými hranicami) pokrývajú celý parametrický priestor v okolí počiatku ϵ, Δ roviny. Prekryv medzi oblasťami stability pevných bodov (ii) a (iii) je artefaktom použitia dvojslučkového priblíženia (v jednoslučkovom priblížení by hranice medzi jednotlivými fázami boli priamky a nedochádzalo by k žiadnemu prekryvu) a môžeme ho chápať ako istý druh hysterézie v systéme. Pre režimy (i),(iii) a (iv) je možné určiť hodnotu parametra priamo z riešení pre nulové body beta funkcií renormalizačnej grupy. Pre režim (ii) rozpadový koeficient má netriviálny tvar v rozvoji po Δ a boli určené jeho prvé tri členy.

  Gaussovský režim odpovedá teórii stredného poľa bez prítomnosti advektívneho poľa. Z fázového diagramu vidíme, že je stabilný pre $\Delta > 0$ (priestorová dimenzia $d > 2$) a vedie k numerickej hodnote $\alpha_G = 1$ predpovedanej kinetickou rovnicou [24]. Kinetický režim II je charakteristický relevantnosťou turbulentných fluktuácii a vidíme, že je stabilný aj v oblasti $\Delta < 0$. Vedie taktiež k rovnakej hodnote exponentu $\alpha_{II} = 1$ ako Gaussovský režim. Preto môžeme konštatovať, že turbulencia vedie k efektívnej homogenizácii systému, ktorá je natoľko silná, že potlačí vplyv difúzie v nízkych rozmeroch. Pre teplotný režim sú turbulentné fluktuácie irelevantné a na pohyb reagujúcich častíc majú vplyv jedine krátkodosahové (teplotné) fluktuácie. Vidíme, že pre tento režim platí $1 + \Delta < \alpha_T < 1$, kde hodnota $\alpha = 1 + \Delta$ je

predpovedaná iba na základe difúzneho pohybu [24] častíc. Teplotné fluktuácie teda vedú k časovému zrýchleniu anihilačného procesu. Nakoniec pre kinetický režim I sú dôležité ako turbulentné tak aj difúzne fluktuácie. Na základe porovnania s inými režimami, je možné ukázať, že platí $\alpha_T < \alpha_I < \alpha_{II}$. Teda turbulentné fluktuácie zvýšia pravdepodobnosť reakcie, ale nie až natoľko, aby sme pozorovali Gausovský (dokonale premiešaný) režim.

Použitím Legendrovej transformácie a podmienky minima efektívneho účinku, bola odvodená renormalizovaná kinetická rovnica so započítaním slučkových korekcií. Z matematického pohľadu sa jedná o nelineárnu integro-diferenciálnu rovnicu, ktorú sme riešili iteračným postupom v rámci ϵ, Δ rozvoja a určili tvar škálovacej funkcie pre strednú koncentráciu častíc pre štyri rôzne IR stabilné režimy. Celkovo možno skonštatovať, že fluktuácie náhodného advektívneho poľa značne ovplyvnia dlhočasové asymptotické správanie sa systému.

- Skonštruovali sme teoreticko-poľový model pre anihilačnú reakciu $A + A \to \varnothing$ v prítomnosti rýchlostných fluktuácii popísaných Kraichnanovým modelom s konečným korelačným časom pre nestlačitelný (dvojslučkové priblíženie) aj pre stlačitelný prípad (jednoslučkové priblíženie).

  Ukázali sme, že stlačiteľnosť má priamy vplyv na hodnotu rozpadového indexu α a vedie obvykle k jeho nárastu. Tento efekt môže byť vysvetlený efektívnym priťahovaním difundujúcich dráh reagujúcich častíc. Ukázalo sa tiež, že v prípade kedy sú relevantné fluktuácie koncentrácie a zároveň aj kompresibilita prostredia, vplyv kompresibility je v porovnaní s uvedenými fluktáciami potlačený.

- Študovali sme možný vplyv náhodných zdrojov a únikových mechanizmov na anihilačnú reakciu $A + A \to \emptyset$. Na rozdiel od zvyčajného postupu, kde sú zdroje a únikové mechanizmy zavedené priamo do Langevinovej rovnice, sme ich zaviedli do riadiacej rovnice.

  Analýzou kanonických dimenzií sme skonštruovali efektívne účinky, ktoré sú základom pre analýzu kritického správania sa študovaných systémov metódou renormalizačnej grupy. Vo všetkých prípadoch sa vplyv náhodných zdrojov a únikových mechanizmov na veľkoškálové správanie sa Greenovských funkcií ukázal ako signifikantný a zmenil triedu univerzality modelu. Namiesto triedy univerzality anihilačného procesu $A + A \to \emptyset$ asymptotické spravanie sa riadí triedou univerzality Gribovského procesu, resp. modifikovaného Gribovského procesu. V prvom prí-

pade je demoštrované, že popis založený na Langevinovej rovnici sa značne líši od postupu založenom na riadiacej rovnici. Náhodný šum v Langevinovej rovnici zodpovedá skôr opisu všade prítomných zdrojov a únikov ako popisu mikroskopických stupňov voľnosti pri prechode k mezoskopickému procesu.

Pre nekritický prípad s náhodným šumom, nezávisiacim na hustote reagujúcich častíc, je Gribovský proces modifikovaný za účelom popisu kritického režimu, kedy zdroje a úbytky vymiznú.

Je potrebná analýza závislosti škálovacích funkcií na parametroch distribučnej funkcie zdrojov a úbytkov v infračervenej limite. Pripomína to situáciu z teórie fázových prechodov, kedy štatistické korelácie parametre usporiadania závisia na "hmotnostnom" člene a závislosť škálovacích funkcií na tejto "hmotnosti" sa študuje.

- Bol skonštruovaný teoreticko-poľový model perkolačného problému v prítomnosti vonkajšieho advektívneho rýchlostného poľa. Po úspešnej renormalizácii boli určené renormalizačné konštanty do prvého rádu poruchovej teórie. Ukázali sme, že systém sa môže v kritickej oblasti nachádzať v jednom z desiatich možných režimov. Prípadu turbulentnej advekcie zodpovedajú dva pevné body:

$$\textbf{FP 5A:} \qquad \frac{g_1^*}{u^*(1+u^*)} = \frac{4\epsilon}{3}, \quad g_2^* = 0 \qquad (10.1)$$
$$\gamma_D = \epsilon = \eta, \quad \gamma_{g_2} = -2\epsilon$$
$$\text{stabilné v oblasti} \qquad \eta = \epsilon > \Delta, \quad \epsilon > 0$$
$$\textbf{FP 5B:} \qquad \frac{g_1^*}{u^*(1+u^*)} = \frac{4(6\epsilon-\Delta)}{15}, \quad g_2 = \frac{8(\Delta-\epsilon)}{5} \qquad (10.2)$$
$$\gamma_D = \epsilon, \quad \gamma_{g_2} = -2\Delta$$
$$\text{stabilné v oblasti} \qquad \epsilon = \eta < \Delta, \quad \epsilon > 0.$$

Tu náboj g_1 odpovedá prítomnosti turbulentných fluktuácií s Kolmogorovským indexom ϵ a u môže byť interpretované ako pomer korelačného času rýchlostných fluktuácii ku korelačnému času skalárneho poľa (poľa koncentrácie reagujúcich častíc) [76]. Náboj g_2 je zodpovedný za kubické interakcie Gribovovských (perkolačných) procesov. Realistická situácia korešpondujúca Kolmogorovskému režimu je dosiahnutý pre hodnoty $\eta = \epsilon = 4/3$, ktoré vedú k známemu "5/3 zákonu" [15] pre priestorové škálovanie rýchlostného poľa. Priamym dosadením zisťujeme, že kritické správanie je popísané fixovaným bodom 5A pre logaritmickú ($\Delta = 0, d = 4$), trojrozmernú ($\Delta = 1/2, d = 3$) a aj dvojrozmernú ($\Delta = 1, d = 2$) situáciu (posledné

dve možnosti je potrebné chápať ako extrapoláciu). Fixovaný bod **5A** je charakteristický nulovou hodnotou náboja g_2. To nás vedie k záveru, že turbulentné fluktuácie potláčajú vplyv nelineárnych perkolačných interakcií a možno ich teda v kritickej oblasti daného režimu zanedbať.

A RESULTS FOR THE FEYNMAN GRAPHS FROM THE CHAPTER 4

In this section we give a list of numerical results for all of the Feynman graphs, that are necessary to an evaluation of renormalization constants Z_2 (4.29) and Z_4 (4.30) to two-loop aproximation.

A.1 FEYNMAM DIAGRAMS FOR PROPAGATOR AND REACTION VERTEX

In the expression for renormalization constants digamma function $\psi(x)$ appear, that is defined as

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x).$$

We use the abberivation with overline for the following expression

$$\overline{f}(z,q) = f(z,q) + f(z,1/q). \tag{A.1}$$

It is convenient to introduce following functions

$$f_1(z,q,u) = \frac{(1-z^2)^{\frac{1}{2}}}{[(1+u)x^2+2-2xz][1+x^2-2xz]}, \quad h_1(z,q) = \frac{(1-z^2)^{\frac{1}{2}}}{[x^2+1-xz][1+x^2-2xz]}, \tag{A.2}$$

$$f_2(z,q,u) = \frac{xz(1-z^2)^{\frac{1}{2}}}{[(1+u)x^2+2-2xz][1+x^2-2xz]}, \quad h_2(z,q) = \frac{xz(1-z^2)^{\frac{1}{2}}}{[x^2+1-xz][1+x^2-2xz]}, \tag{A.3}$$

$$f_3(z,q,u) = \frac{z^2(1-z^2)^{\frac{1}{2}}}{[(1+u)x^2+2-2xz][1+x^2-2xz]}, \quad h_3(z,q) = \frac{z^2(1-z^2)^{\frac{1}{2}}}{[x^2+1-xz][1+x^2-2xz]}, \tag{A.4}$$

$$f_4(z,q,u) = \frac{x^2(1-z^2)^{\frac{1}{2}}}{[(1+u)x^2+2-2xz][1+x^2-2xz]}, \quad h_4(z,q) = \frac{x^2(1-z^2)^{\frac{1}{2}}}{[x^2+1-xz][1+x^2-2xz]}, \tag{A.5}$$

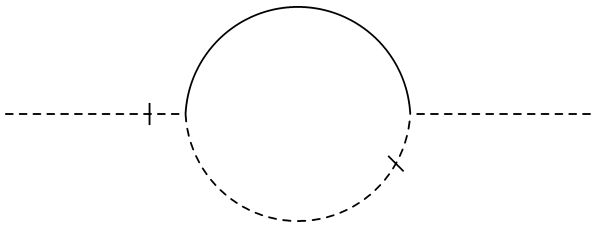
$$f_5(z,q,u) = \frac{x^2z^2(1-z^2)^{\frac{1}{2}}}{[(1+u)x^2+2-2xz][1+x^2-2xz]}, \quad h_5(z,q) = \frac{x^2z^2(1-z^2)^{\frac{1}{2}}}{[x^2+1-xz][1+x^2-2xz]}, \tag{A.6}$$

$$f_6(z,q,u) = \frac{(1-z^2)^{\frac{3}{2}}}{[(1+u)x^2+2-2xz][1+x^2-2xz]}, \quad h_6(z,q) = \frac{(1-z^2)^{\frac{3}{2}}}{[x^2+1-xz][1+x^2-2xz]}, \tag{A.7}$$

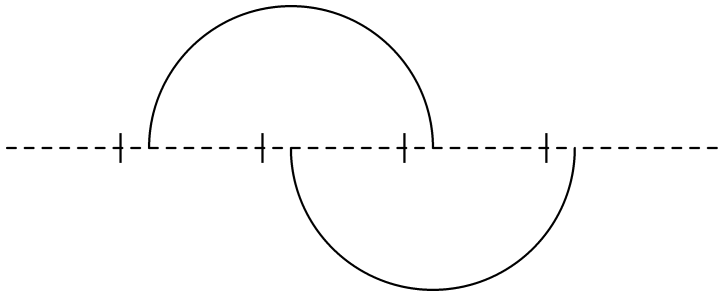
$$f_7(z,q,u) = \frac{x^2(1-z^2)^{\frac{3}{2}}}{[(1+u)x^2+2-2xz][1+x^2-2xz]^2}, \quad h_7(z,q) = \frac{x^2(1-z^2)^{\frac{3}{2}}}{[x^2+1-xz][1+x^2-2xz]^2}, \tag{A.8}$$

$$g_1(z,q,u) = \frac{x^3z(1-z^2)^{\frac{1}{2}}}{[(1+u)x^2+2-2xz][1+x^2-2xz]}, \tag{A.9}$$

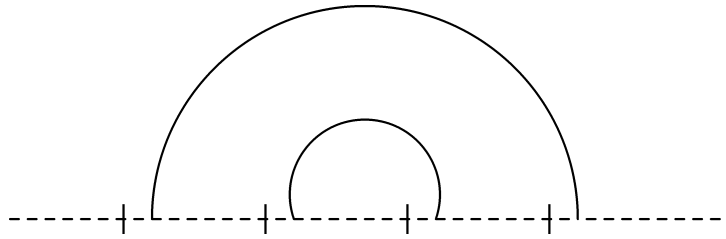
Dyson equation for the propagator $\langle \psi \psi^{+} \rangle$ could be schematically written as Symbolic computation of the Feynman graphs to the two-loop order yields the following results



$$-\overline{S_2}\frac{Dp^2}{4u(1+u)}\left\{\frac{g_1}{2\epsilon}\left(\frac{\mu}{m}\right)^{2\epsilon}-\frac{g_2}{2\Delta}\left(\frac{m}{\mu}\right)^{2\Delta}\right\} \tag{A.10}$$



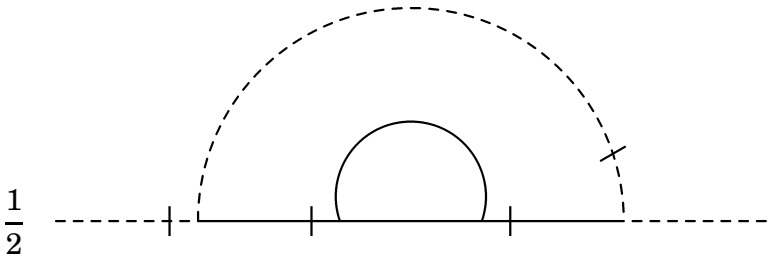
$$\frac{\overline{S_2}^2}{64}\,\frac{Dp^2}{u^2(1+u)^2}\left(1+\frac{1+2u}{u^2}\ln\frac{1+2u}{(1+u)^2}\right)\left\{\frac{g_1^2}{4\epsilon}\left(\frac{\mu}{m}\right)^{4\epsilon}-\frac{g_2^2}{4\Delta}\left(\frac{m}{\mu}\right)^{4\Delta}+\frac{2g_1g_2}{2(\epsilon-\Delta)}\left(\frac{\mu}{m}\right)^{2\epsilon-2\Delta}\right\} \tag{A.11}$$



$$\frac{\overline{S_2}^2 Dp^2}{32u(1+u)^3}\,\frac{g_1^2}{4\epsilon}\Big(\frac{\mu}{m}\Big)^{4\epsilon}\Big[\frac{1}{\epsilon}+2\xi+1+\frac{1+2u}{u^2}\ln\frac{1+2u}{(1+u)^2}\Big] \tag{A.12}$$

$$\frac{\overline{S_2}^2 Dp^2}{32u(1+u)^3}\,\frac{g_2^2}{4\Delta}\Big(\frac{m}{\mu}\Big)^{4\Delta}\Big[\frac{1}{\Delta}+1-\frac{1+2u}{u^2}\ln\frac{1+2u}{(1+u)^2}\Big] \tag{A.13}$$

$$\frac{\overline{S_2}^2 Dp^2}{32u(1+u)^3}\,\frac{g_1 g_2}{2(\epsilon-\Delta)}\Big(\frac{\mu}{m}\Big)^{2\epsilon-2\Delta}\Big[\frac{1}{\epsilon}-\frac{1}{\Delta}+2\xi+2\frac{1+2u}{u^2}\ln\frac{1+2u}{(1+u)^2}\Big] \tag{A.14}$$



$$\frac{\overline{S_2}^2}{128}\,\frac{Dp^2}{u(u^2-1)}\,\frac{g_1^2}{4\epsilon}\Big(\frac{\mu}{m}\Big)^{4\epsilon}\Big\{\frac{1-u}{2\epsilon+\Delta}+(1-u)\frac{3\xi}{2+\xi}+\frac{16}{\pi}\int_1\frac{dq}{q}\int_{-1}^{1}dzI(z,q)\Big\} \tag{A.15}$$

$$\frac{\overline{S_2}^2}{64}\,\frac{Dp^2}{u(u^2-1)}\,\frac{g_1 g_2}{2(\epsilon-\Delta)}\Big(\frac{\mu}{m}\Big)^{2\epsilon-2\Delta}\Big\{\frac{1-u}{\epsilon}+3\xi(1-u)+\frac{16}{\pi}\int_1\frac{dq}{q}\int_{-1}^{1}dzI(z,q)\Big\} \tag{A.16}$$

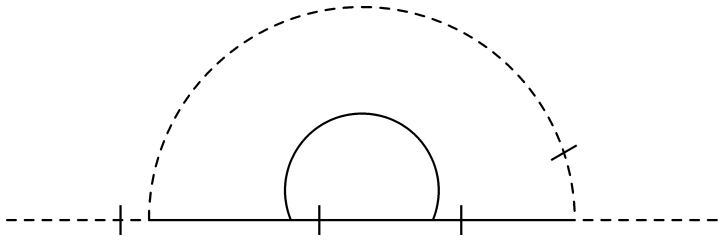
$$\frac{\overline{S_2}^2}{128}\,\frac{Dp^2}{u(u^2-1)}\,\frac{g_2^2}{4\Delta}\Big(\frac{m}{\mu}\Big)^{4\Delta}\Big\{\frac{1-u}{\Delta}+3(1-u)-\frac{16}{\pi}\int_1\frac{dq}{q}\int_{-1}^{1}dzI(z,q,u)\Big\} \tag{A.17}$$

where

$$I(z,q,u) = 2F(z,q,u) - uH(z,q), \tag{A.18}$$

$$F(z,q,u) = -\overline{f_2}(z,q,u) - \overline{f_6}(z,q,u) + f_6(z,0,u) + \frac{\overline{f_4}(z,q,u)}{4} + \overline{f_1}(z,q,u) - f_1(z,0,u)$$

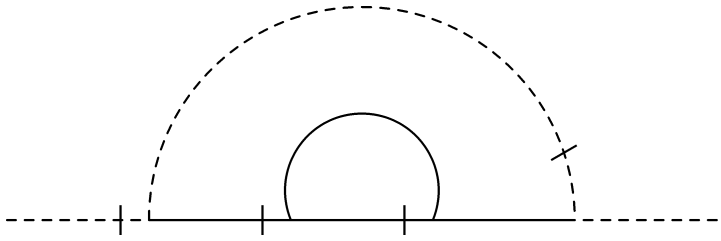
$$H(z,q) = -\overline{h_2}(z,q) + \frac{\overline{h_4}(z,q)}{4} - \overline{h_6}(z,q) + h_6(z,0) + \overline{h_1}(z,q) - h_1(z,0) \tag{A.19}$$



$$\frac{\overline{S_2^2}}{256u(1+u)}\, Dp^2 \frac{g_1^2}{4\epsilon}\left(\frac{\mu}{m}\right)^{4\epsilon}\left[\frac{1}{\epsilon} + 3\xi + \ln 4/3 - 16\mathrm{arctgh}1/2 + 16\ln 2 - \frac{7}{2}\right] \tag{A.20}$$

$$\frac{\overline{S_2^2}}{256u(1+u)}\, Dp^2 \frac{g_2^2}{4\Delta}\left(\frac{m}{\mu}\right)^{4\Delta}\left[\frac{1}{\Delta} - \ln 4/3 + 16\mathrm{arctgh}1/2 - 16\ln 2 + \frac{13}{2}\right] \tag{A.21}$$

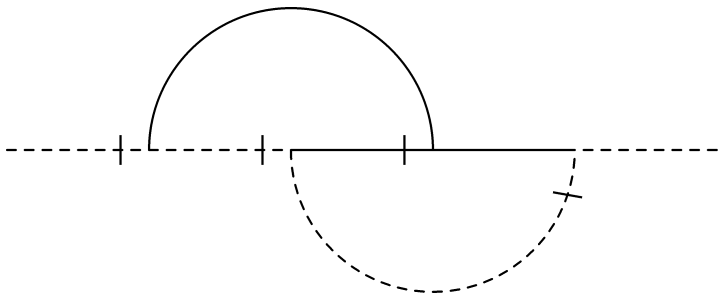
$$\frac{\overline{S_2^2}}{256u(1+u)}\, Dp^2 \frac{g_1 g_2}{2(\epsilon-\Delta)}\left(\frac{\mu}{m}\right)^{2\epsilon-2\Delta}\left[\frac{1}{\epsilon} - \frac{1}{\Delta} + 3\xi + 2\ln 4/3 - 32\mathrm{arctgh}1/2 + 32\ln 2 - 10\right] \tag{A.22}$$



$$\frac{\overline{S_2^2}Dp^2}{256u(u-1)}\,\frac{g_1^2}{4\epsilon}\left(\frac{\mu}{m}\right)^{4\epsilon}\Big[\frac{u^2+2u-3}{(1+u)^2}\,\left(\frac{1}{\epsilon}+\xi\right)+16\ln 2-16\mathrm{arctgh}1/2+\ln 4/3-\frac{7}{2}-$$
$$\frac{12}{(1+u)^2}\ln\frac{2}{1+u}-\frac{8}{1+u}-\frac{6}{(1+u)^2}-\frac{16u^2+48u+20}{(1+u)^2}\ln\frac{1+2u}{2+2u}-$$
$$\frac{128}{(1+u)^2\pi}\int_1\frac{dk}{k}\int_{-1}^{1}dzF_D(z,q,u)\Big] \tag{A.23}$$
$$\frac{\overline{S_2^2}Dp^2}{256u(u-1)}\,\frac{g_2^2}{4\Delta}\left(\frac{m}{\mu}\right)^{4\Delta}\Big[\frac{u^2+2u-3}{(1+u)^2}\,\left(\frac{1}{\Delta}+1\right)-16\ln 2+16\mathrm{arctgh}1/2-\ln 4/3+\frac{7}{2}+$$
$$\frac{12}{(1+u)^2}\ln\frac{2}{1+u}+\frac{8}{1+u}+\frac{6}{(1+u)^2}+\frac{16u^2+48u+20}{(1+u)^2}\ln\frac{1+2u}{2+2u}+$$
$$\frac{128}{(1+u)^2\pi}\int_1\frac{dk}{k}\int_{-1}^{1}dzF_D(z,q,u)\Big] \tag{A.24}$$
$$\frac{\overline{S_2^2}Dp^2}{256u(u-1)}\,\frac{g_1g_2}{2(\epsilon-\Delta)}\left(\frac{\mu}{m}\right)^{2\epsilon-2\Delta}\Big[\frac{u^2+2u-3}{(1+u)^2}\,\left(\frac{1}{\epsilon}-\frac{1}{\Delta}+\xi-1\right)+32\ln 2-32\mathrm{arctgh}1/2+$$
$$2\ln 4/3-7-\frac{24}{(1+u)^2}\ln\frac{2}{1+u}-\frac{16}{1+u}-\frac{12}{(1+u)^2}-\frac{32u^2+96u+40}{(1+u)^2}\ln\frac{1+2u}{2+2u}-$$
$$\frac{256}{(1+u)^2\pi}\int_1\frac{dk}{k}\int_{-1}^{1}dzF_D(z,q,u)\Big] \tag{A.25}$$

where

$$F_D(z,q,u)=-\overline{f}_1(z,k)+f_1(z,0)-\overline{f}_2(z,k)+2\overline{f}_3(z,k)-2f_3(z,0) \tag{A.26}$$



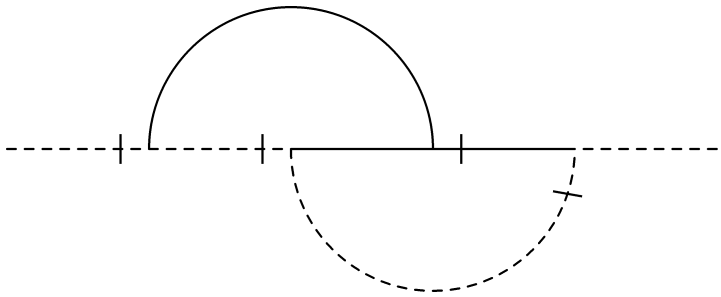
$$\frac{Dp^2\overline{S_2^2}}{16u(1+u)^2(1-u)}\ \frac{g_1^2}{4\epsilon}\Big(\frac{\mu}{m}\Big)^{4\epsilon}\Big[1+(1+2u)\ln\frac{1+2u}{2+2u}+\frac{8}{\pi}\int_1^\infty\frac{dk}{k}\int_{-1}^1 F_B(z,q,u)\Big] \quad (A.27)$$

$$-\frac{Dp^2\overline{S_2^2}}{16u(1+u)^2(1-u)}\ \frac{g_2^2}{4\Delta}\Big(\frac{m}{\mu}\Big)^{4\Delta}\Big[1+(1+2u)\ln\frac{1+2u}{2+2u}+\frac{8}{\pi}\int_1^\infty\frac{dk}{k}\int_{-1}^1 F_B(z,q,u)\Big] \quad (A.28)$$

$$\frac{p^2D\overline{S_2^2}}{8u(1+u)^2(1-u)}\ \frac{g_1g_2}{2(\epsilon-\Delta)}\Big(\frac{\mu}{m}\Big)^{2\epsilon-2\Delta}\Big[1+(1+2u)\ln\frac{1+2u}{2+2u}+\frac{8}{\pi}\int_1^\infty\frac{dk}{k}\int_{-1}^1 F_B(z,q,u)\Big] \quad (A.29)$$

where

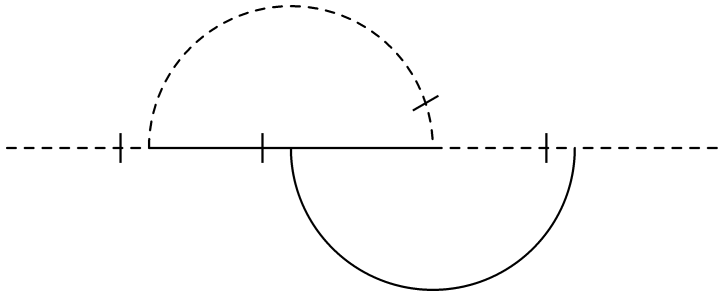
$$F_B(z,q,u)=\overline{f_5}(z,q,u)-\overline{f_2}(z,q,u) \quad (A.30)$$



$$\frac{\overline{S_2^2}Dp^2}{64u(1-u^2)}\ \frac{g_1^2}{4\epsilon}\Big(\frac{\mu}{m}\Big)^{4\epsilon}\Big[8\mathrm{arctgh}1/2+(u-1)(4-2\gamma-2\psi(3/2))-5\ln 4/3-$$
$$4(1+u)\mathrm{arctgh}\frac{u}{u+1}-\frac{2u^2+2u+1}{u}\ln\frac{1+2u}{(1+u)^2}\Big] \quad (A.31)$$

$$-\frac{\overline{S_2^2}Dp^2}{64u(1-u^2)}\ \frac{g_2^2}{4\Delta}\Big(\frac{m}{\mu}\Big)^{4\Delta}\Big[8\mathrm{arctgh}1/2+(u-1)(4-2\gamma-2\psi(3/2))-5\ln 4/3-$$
$$4(1+u)\mathrm{arctgh}\frac{u}{u+1}-\frac{2u^2+2u+1}{u}\ln\frac{1+2u}{(1+u)^2}\Big] \quad (A.32)$$

$$\frac{\overline{S_2^2}Dp^2}{32u(1-u^2)}\ \frac{g_1g_2}{2(\epsilon-\Delta)}\Big(\frac{\mu}{m}\Big)^{2\epsilon-2\Delta}\Big[8\mathrm{arctgh}1/2+(u-1)(4-2\gamma-2\psi(3/2))-5\ln 4/3-$$
$$4(1+u)\mathrm{arctgh}\frac{u}{u+1}-\frac{2u^2+2u+1}{u}\ln\frac{1+2u}{(1+u)^2}\Big] \quad (A.33)$$



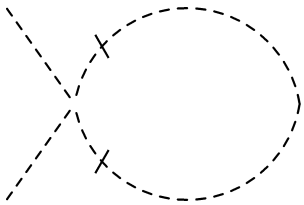
$$-Dp^2\frac{\overline{S_d}^2}{8\pi u(1+u)}\frac{g_1^2}{4\epsilon}\left(\frac{\mu}{m}\right)^{4\epsilon}\int_1\frac{dq}{q}\int_{-1}^{1}dzM(z,q,u) \tag{A.34}$$

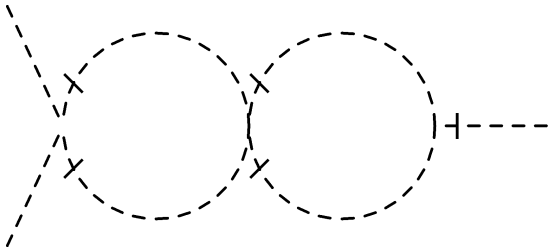
$$Dp^2\frac{\overline{S_d}^2}{8\pi u(1+u)}\frac{g_2^2}{4\Delta}\left(\frac{m}{\mu}\right)^{4\Delta}\int_1\frac{dq}{q}\int_{-1}^{1}dzM(z,q,u) \tag{A.35}$$

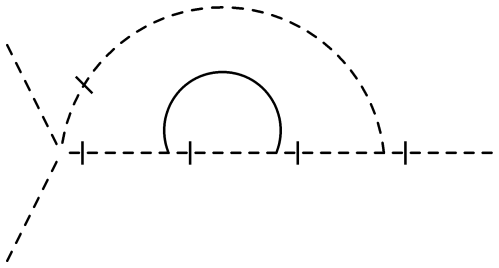
$$-Dp^2\frac{\overline{S_d}^2}{4\pi u(1+u)}\frac{g_1g_2}{2(\epsilon-\Delta)}\left(\frac{\mu}{m}\right)^{2\epsilon-2\Delta}\int_1\frac{dq}{q}\int_{-1}^{1}dzM(z,q,u) \tag{A.36}$$

where

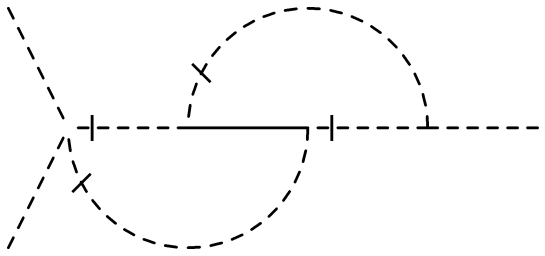
$$M(z,q,u)=\frac{2\overline{h}_5(z,q)-\overline{h}_2(z,q)}{2(u-1)}+\frac{2\overline{f_5}(z,q,u)-\overline{g}_1(z,q,u)}{1+u}+\frac{4\overline{f}_5(z,q,u)-2\overline{f}_2(z,q,u)}{1-u^2} \tag{A.37}$$

$\frac{1}{2}$  $= -2\overline{S_2}\frac{D\lambda^2}{\Delta}\left(\frac{m}{\mu}\right)^{2\Delta}$ (A.38)

$\frac{1}{4}$  $= -\overline{S_2}^2 \frac{D\lambda^3}{\Delta^2}\left(\frac{m}{\mu}\right)^{4\Delta}$ (A.39)

 =

$$-D\lambda^2 \frac{\overline{S_2}^2}{2u(1+u)} \frac{g_1}{2(\epsilon-\Delta)}\left(\frac{\mu}{m}\right)^{2\epsilon-2\Delta}\left[\frac{1}{\epsilon}+\xi+1+\ln\frac{1+u}{2u}+\frac{u+2}{u}\ln\frac{2+u}{2+2u}\right]$$
$$-D\lambda^2 \frac{\overline{S_2}^2}{2u(1+u)} \frac{g_2}{4\Delta}\left(\frac{m}{\mu}\right)^{4\Delta}\left[\frac{1}{\Delta}-\ln\frac{1+u}{2u}-\frac{u+2}{u}\ln\frac{2+u}{2+2u}\right] \quad \text{(A.40)}$$

$\frac{1}{2}$  =

$$\frac{\overline{S_2^2}}{\pi u}\lambda^2 D\left[\frac{g_1}{2(\epsilon-\Delta)}\left(\frac{\mu}{m}\right)^{2\epsilon-2\Delta}-\frac{g_2}{4\Delta}\left(\frac{m}{\mu}\right)^{4\Delta}\right]\int_{-1}^{1} dz(1-z^2)^{\frac{d-1}{2}} I(z,u), \quad \text{(A.41)}$$

where

$$I(z,u) = \frac{2}{(1-u)^2+4uz^2}\Big\{\frac{u-1}{2}\ln\frac{2u}{1+u} - \frac{2(1+u)z}{\sqrt{1-z^2}}\Big[\frac{\pi}{2} - \arctan\sqrt{\frac{1+z}{1-z}}\Big] + \frac{u(u+3)z}{\sqrt{2u(1+u)-u^2z^2}}\Big[\pi - \arctan\frac{zu+u+1}{\sqrt{2u(1+u)-u^2z^2}} - \arctan\frac{(2+z)u}{\sqrt{2u(1+u)-u^2z^2}}\Big]\Big\} \tag{A.42}$$

A.2 RENORMALIZATION CONSTANTS AND FIXED POINTS

Isolating the terms containing poles in the equations (4.25) and (4.27) it is possible to calculate all contributions to the renormalization constant Z_2

$$A_{11} = -\frac{(1+\xi)u^2+(3\xi+2)u+6\xi+1}{512u(1+u)^3\xi}, \quad A_{12} = -\frac{(1+\xi)u^2+(1+3\xi)u+6\xi-4}{256u(1+u)^3(1-\xi)}, \tag{A.43}$$

$$A_{22} = -\frac{u+5}{512u(1+u)^3}, \tag{A.44}$$

$$B_{11} = B_1(u)+B_2(u,\xi), \quad B_{12} = 4[B_1(u)+B_3(u,\xi)], \quad B_{22} = -B_1(u)-B_2(u,-1), \tag{A.45}$$

where the functions $B_1(u), B_2(u,\xi)$ and $B_3(u,\xi)$ are given by the expressions

$$B_1(u) = \frac{1}{1024u^4(1+u)^3(u-1)}\Big[-12u^3(1+u)\ln\frac{2}{1+u} + 32u^4(1+u)^2\ln 2 + 2u^3(1+u)^2\times (u+10)\ln\frac{4}{3} - 32u^3(1+u)^3\operatorname{arctgh}\frac{1}{2} + 4(2u^6+6u^5+7u^4+10u^3-4u-1)\times \ln\frac{1+2u}{(1+u)^2} - 4u^3(1+u)[4u^2+20u+9]\ln\frac{1+2u}{2+2u} + 16u^3(1+u)^3\operatorname{arctgh}\frac{u}{u+1} + 8u^3(1+u)^2(u-1)(\gamma+\psi(3/2)) - u^2(23u^4+38u^3+17u^2+22u+4)\Big] - \frac{1}{128\pi u(1+u)^2(u-1)}\int_1^\infty dq\int_{-1}^1 dz F(z,q,u), \tag{A.46}$$

$$B_2(u,\xi) = \xi\frac{(2+4\xi)u^2+(10\xi+8)u+38+22\xi}{1024u(1+u)^3(2+\xi)}, \tag{A.47}$$

$$B_3(u,\xi) = -\frac{(8\xi+4)u^2+(14\xi+10)u+22-10\xi}{1024u(1+u)^3}, \tag{A.48}$$

where the function $F(z,q,u)$

$$
\begin{aligned}
F(z,x,u) &= (1-z^2)^{1/2}\frac{M(z,x,u)}{N(z,x,u)} \qquad \text{(A.49)}\\
M(z,x,u) &= (x^6+1)[z^3(24u^3+24u^2+72u+72)-z(8u^3+12u^2+8u+60)]+(x^5+x)\times\\
&\quad [-z^4(40u^3+88u^2+120u+264)+z^2(-4u^3+16u^2+108u+168)+4u^3+\\
&\quad 14u^2+28u+18]+(x^4+x^2)[z^5(16u^3+96u^2+48u+288)+z^3(12u^4+64u^2+\\
&\quad 128u^2+96u+180)-z(4u^4+26u^3+92u^2+174u+312)]+x^3[-z^6(32u^2+\\
&\quad 96)-z^4(8u^4+64u^3+240u^2+144u+600)+z^2(-8u^4+4u^3+84u^2+108u+\\
&\quad 452)+2u^4+6u^3+26u^2+58u+36],\\
N(z,x,u) &= (1+x^2-2xz)(1+x^2-xz)((1+u)x^2+2-2xz)(1+u+2x^2-2xz) \qquad \text{(A.50)}
\end{aligned}
$$

has been introduced. The function $C(u,\xi)$ in (4.30) resulting from the expression (4.28) is given as

$$
C(u,\xi)=-\frac{1}{8u\pi}\int_{-1}^{1}dz(1-z^2)^{\frac{1}{2}}G(z,u)+\frac{1}{8u(1+u)}\Big(\ln\frac{1+u}{2u}+1+\frac{2+u}{u}\ln\frac{u+2}{2u+2}+\xi\Big), \qquad \text{(A.51)}
$$

where

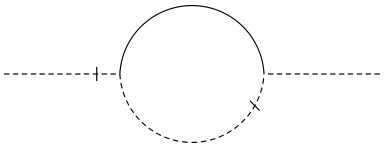
$$
\begin{aligned}
G(z,u) &= \frac{4}{(1-u)^2+4uz^2}\Big\{\frac{u-1}{2}\ln\frac{2u}{1+u}-\frac{2(1+u)z}{\sqrt{1-z^2}}\Big[\frac{\pi}{2}-\arctan\sqrt{\frac{1+z}{1-z}}\Big]+\\
&\quad \frac{u(u+3)z}{\sqrt{2u(1+u)-u^2z^2}}\Big[\pi-\arctan\frac{zu+u+1}{\sqrt{2u(1+u)-u^2z^2}}-\arctan\frac{(2+z)u}{\sqrt{2u(1+u)-u^2z^2}}\Big]\Big\}. \qquad \text{(A.52)}
\end{aligned}
$$

Solving equations (4.37) for the fixed points (4.48) and (4.51) for the coordinate u^* yields the second order correction in the form

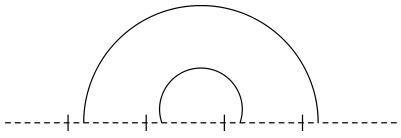
$$
\begin{aligned}
u_1^*(\xi) &= \frac{8R}{3\sqrt{17}}-\frac{8192}{3\sqrt{17}}B_1(u_0^*)-\frac{1}{432\sqrt{17}(1+\xi)^2(2+\xi)}\Big((21384-648\sqrt{17})\xi^4+\\
&\quad (52512-2592\sqrt{17})\xi^3+(22192-2736\sqrt{17})\xi^2+(72\sqrt{17}-29064)\xi+720\sqrt{17}-18768\Big)\\
g_{12}^*(\xi) &= 64\frac{R(2+3\xi)-1}{27(1+\xi)}-16\frac{2+3\xi}{243(1+\xi)^4(2+\xi)}\Big[45\xi^4+213\xi^3+349\xi^2+231\xi+50\Big]\\
g_{22}^*(\xi) &= 64\frac{1+R}{27(1+\xi)}-16\frac{2+3\xi}{243(1+\xi)^4(2+\xi)}\Big[57\xi^4+171\xi^3+185\xi^2+93\xi+22\Big]. \qquad \text{(A.53)}
\end{aligned}
$$

where $R = -0.168$ is numerical constant and $B_1(u_0^*)$ is value of the function $B_1(u)$ introduced in (A.48) for the first order correction $u_0^* = (\sqrt{17}-1)/2$ to the value of u^*.

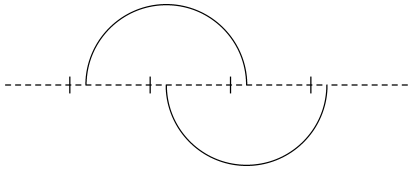
B RESULTS FOR THE FEYNMAN GRAPHS FROM THE CHAPTER 5



$$= -\frac{gDp^2\overline{S_2}}{8u(1+u)\epsilon}\left(\frac{\mu}{m}\right)^{2\epsilon} \tag{B.1}$$



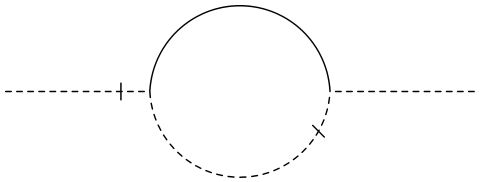
$$= \frac{g^2Dp^2\overline{S_2}^2}{256u^2(1+u)^2\epsilon}\left(\frac{\mu}{m}\right)^{4\epsilon}\left[1+(u^2+2u)\ln\frac{u^2+2u}{(1+u)^2}\right] \tag{B.2}$$



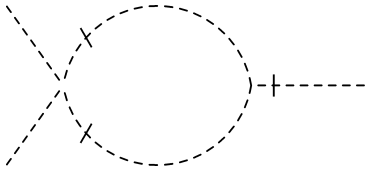
$$= \frac{g^2Dp^2\overline{S_2^2}}{128u^2(1+u)^3\epsilon}\left(\frac{\mu}{m}\right)^{4\epsilon}\left[\frac{1}{\epsilon}+2\xi+1+(u^2+2u)\times\right.$$

$$\left.\ln\frac{u^2+2u}{(1+u)^2}\right] \tag{B.3}$$

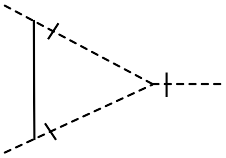
C RESULTS FOR THE FEYNMAN GRAPHS FROM THE CHAPTER 6



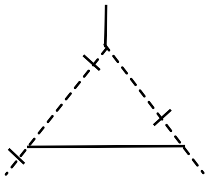
$$= \frac{gD\overline{S_d}p^2}{4ud\epsilon}\left(\frac{\mu}{m}\right)^{2\epsilon}\left[-\frac{d-1+\alpha}{1+u}+\frac{2\alpha}{(1+u)^2}\right] \quad \text{(C.1)}$$

$\frac{1}{2}$ 

$$= -\frac{2D\lambda^2\overline{S_d}}{\Delta}\left(\frac{m}{\mu}\right)^{2\Delta} \quad \text{(C.2)}$$

$\frac{1}{2}$ 

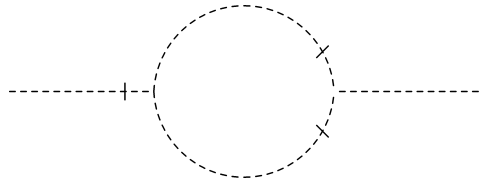
$$= -\frac{\alpha g\lambda D\overline{S_d}}{2u(1+u)\epsilon}\left(\frac{\mu}{m}\right)^{2\epsilon} \quad \text{(C.3)}$$

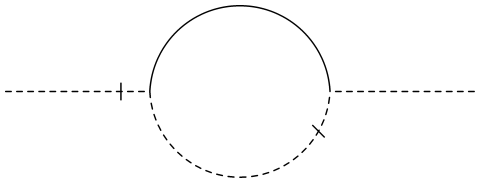


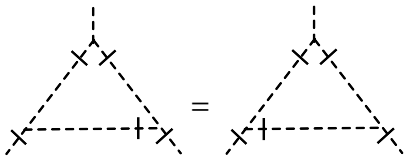
$$= -\frac{i\mathbf{p}\alpha g\overline{S_d}}{4u(1+u)^2d\epsilon}\left(\frac{\mu}{m}\right)^{2\epsilon}, \quad \text{(C.4)}$$

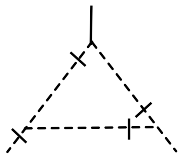
where $\mathbf{p}$ is incoming momentum of the field $\psi^\dagger$.

D RESULTS FOR THE FEYNMAN GRAPHS FROM THE CHAPTER 8

$\frac{1}{2}$  $= \frac{g_2 D \overline{S_d}}{8\Delta}\left(\frac{\mu}{m}\right)^{2\Delta}\left[\tau - \frac{i\omega}{2D} + p^2\frac{d-2}{2d}\right]$ (D.1)

 $= -\frac{g_1 D p^2 \overline{S_d}}{4u(1+u)\epsilon}\frac{d-1}{d}\left(\frac{\mu}{m}\right)^{2\epsilon}$ (D.2)

 $= -\frac{g_2 \overline{S_d}}{8\Delta}\left(\frac{\mu}{m}\right)^{2\Delta}\lambda D \mu^{\Delta}$ (D.3)

 $= -\frac{i\mathbf{p} g_2 \overline{S_d}(d-2)}{8d\Delta}\left(\frac{\mu}{m}\right)^{2\Delta}$ (D.4)

where $\mathbf{p}$ is incoming momentum of the field $\psi^{\dagger}$.

E LIST OF PUBLICATIONS

(1) Strečka J., Čanová L., **Lučivjanský T.**, Jaščur M., Multiple frustration-induced plateaus in a magnetization process of the mixed spin-1/2 and spin-3/2 Ising-Heisenberg diamond chain, J. Phys .: Conf. Ser. **145**, 012058 (2009).

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(3) M. Hnatič, **T. Lučivjanský**: Study of Anomalous Kinetics of the Reaction A+A-> 0 to the Second Order of the Perturbation Scheme, The 11th Small Triangle Meeting on theoretical physics, Kysak, 20-23 September 2009, Slovakia, ISBN 978-80-968060-8-9

(4) **T. Lučivjanský**, M. Hnatič, J. Honkonen: Effective Field Theoretic Model For The Special Class Of Chemical Reactions, 18th Conference of Slovak Physicists, Banská Bystrica, 6-9 September 2010, Slovakia, ISBN 978-80-970625-0-7

(5) M. Hnatič, J. Honkonen, **T. Lučivjanský** : Anomalous Kinetics of the Annihilation Process in the Random Environment, The 12th Small Triangle Meeting on theoretical physics, Stakčín, 19-22 September 2010, ISBN 978-80-89284-88-7

(6) M. Hnatič, J. Honkonen, **T. Lučivjanský** : Field Theory Approach In Kinetic Reaction: Role Of Random Sources And Sinks, Theor. Math. Phys. **169**(1) 1489-1498 (2011)

(7) M. Hnatič, J. Honkonen, **T. Lučivjanský** : Study of Anomalous Kinetics of The Annihilation Reaction A+A->0, Theor. Math. Phys. **169**(1) 1481-1488 (2011)

(8) M. Hnatič, J. Honkonen, **T. Lučivjanský** : On the Mathematical Modelling of the Annihilation Process, Mathematical Modelling and Computational Physics Stará Lesná 2011, Lecture Notes in Computer Science 7125, (Springer-Verlag) ISBN 978-3-642-28211-9

(9) M. Hnatič, **T. Lučivjanský**, Critical Behaviour Of Directed Percolation In The Presence Of Synthetic Velocity Field , The 13th Small Triangle Meeting on theoretical physics, Stará Lesná, 13-16 November 2012, ISBN 978-80-8143-017-6

(10) M. Hnatič, **T. Lučivjanský**, M. Dančo, L. Mižišin: Influence Of The Compressibility On The Anomalous Kinetics Of The Annihilation Process, book of abstracts for the 37th Conference Of The Middle European Cooperation in Statistical Physics, Tatranské Matliare 2012

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